Overview of the Poisson Process

1 Definition of a counting process.

A counting process, $N(t)$, is a representation of the total number of events up to the time t. Counting processes have the following properties:

1. $N(t) \geq 0$
2. $N(t)$ is integer valued.
3. If $s < t$ then $N(s) \leq N(t)$. In other words, $N(t)$ is non-decreasing.
4. If $s < t$, then $N(t) - N(s)$ are the number of events in $(s, t]$.

2 Definition of independent increments.

A stochastic process has independent increments if $N(t) - N(s)$ and $N(u) - N(v)$ are independent for any non-overlaping intervals $(s, t]$ and $(v, u]$.

3 Definition of stationary increments.

A stochastic process has stationary increments if the distribution of $N(t) - N(s)$ depends only on the length of the interval $(s, t]$ and not on the location of that interval.

4 First Definition of The Poisson Process

The process, $N(t)$, is a Poisson process with rate λ if it is a counting process and

1. $N(0) = 0$.
2. The process has independent increments.

3. $P(N(t+h) - N(t) = 1) = \lambda h + o(h)$.
4. $P(N(t+h) - N(t) \geq 2) = o(h)$

5 Second Definition of The Poisson Process

The process, $N(t)$, is a Poisson process with rate λ if it is a counting process and

1. $N(0) = 0$.
2. The process has independent increments.
3. $P(N(t+s) - N(s) = n) = e^{-\lambda t} \frac{(\lambda t)^(n)}{n!}$

Note: We showed that the second definition can be proven from the first using differential equations.

6 Third Definition of The Poisson Process

Given a sequence of iid exponential random variables each with mean $1/\lambda$, $T_1, T_2, ...$, then we can define a Poisson process as

$$N(t) = \max\{n : S_n \leq t\}$$

where $S_n = \sum_{i=1}^{n} T_i$ and $S_0 = 0$.
Note: This definition was shown to be a consequence of the second when we showed that using the second definition implies exponential and independent interarrival times. From this definition, we also note that the distribution of the time to the nth arrival has a Gamma distribution with mean n/λ and variance n/λ^2.

7 Location of Events

The density for the location of events conditioned on there being n events up to time t is the same as the density for the order statistics of a random sample of n observations from a uniform distribution from zero to t. If we denote the location (in time) of the n events as $S_1, S_2, ..., S_n$, then the conditional density of these random variables given $N(t) = n$ is given as

$$f(s_1, s_2, ..., s_n | N(t) = n) = \frac{n!}{t^n}, \quad s_1 < s_2 < s_3 < ... < s_n$$

8 Superposition of Poisson Processes

Summing m independent Poisson processes, $N_1(t), N_2(t), ..., N_m(t)$ with rates $\lambda_1, \lambda_2, ..., \lambda_m$ yields a Poisson process, $N(t)$ with rate $\lambda = \lambda_1 + ... + \lambda_m$.

9 Thinning of a Poisson Process

If events occur following a Poisson process, $N(t)$, with rate λ, and at each occurrence an event is assigned independently to one of m categories with probabilities $p_1, p_2, ..., p_m$, then the process describing the arrivals for the ith category follow a Poisson process with parameter $p_i\lambda$ and is independent of the other processes.

For only two categories, we may define independent Bernoulli random variables $X_1, ..., X_n, ...$ which are one when an event is assigned to category one with probability p_1 and zero when an event is assigned to category 2 with probability $p_2 = 1 - p_1$. If $N(t)$ is a Poisson process with parameter λ, then

$$N_1(t) = \sum_{i=1}^{N(t)} X_i$$

is a Poisson process with parameter $p_1\lambda$ and is independent of the number of events assigned to category two at time t,

$$N_2(t) = N(t) - \sum_{i=1}^{N(t)} X_i$$

which is a Poisson process with parameter $(1 - p_1)\lambda$.

10 First Definition of Nonhomogeneous Poisson Process

$N(t)$ is a nonhomogeneous Poisson process with arrival rate $\lambda(t)$ if it is a counting process such that

1. The increments are independent.
2. $N(0) = 0$.
3. The probability of more than one event is negligible. $P(N(t+h) - N(t) \geq 2) = o(h)$.

4. The probabiity of one event in a small interval is $P(N(t+h)-N(t)=1)=\lambda(t)h+o(h)$.

11 Second Definition of Nonhomogeneous Poisson Process

$N(t)$ is a nonhomogeneous Poisson process with arrival rate $\lambda(t)$ if it is a counting process such that

1. The increments are independent.
2. $N(0)=0$.
3. $P(N(v)-N(u)=n)=\frac{\left(\int_u^v \lambda(t)dt\right)^n}{n!}e^{-\int_u^v \lambda(t)dt}$

12 Central Limit Theorem for a Poisson Process

For a Poisson process, $N(t)$ with parameter λ,

$$\lim_{t\to\infty} P\left(\frac{N(t)-\lambda t}{\sqrt{\lambda t}} \leq x\right) = \Phi(x)$$

where $\Phi(x)$ is the cdf for a standard normal distribution.
Note: This was shown by verifying that the moment generating function for the scaled Poisson process converges to the moment generating function for a standard normal random variable.