STAT 514
Homework 1

Due 1.20.2011

1. Suppose that we want to generate the outcome of the flip of a fair coin, but that all we have at our disposal is a biased coin which lands on heads with some unknown probability p that need not be equal to 1/2. Consider the following algorithm for accomplishing our task.

   (a) Flip the coin.

   (b) Flip the coin agian.

   (c) If both flips land on heads or both land on tails, return to step (a).

   (d) Let the result of the last flip be the result of the algorithm.

   Now, answer the following: (Note that it is great if you have intuition about why this is true; however, show the answers using the calculus of probability.)

   (a) Show that the result is equally likely to be heads or tails.

   (b) What are the mean and variance of the number of total flips required to perform this algorithm?

2. Let $U_1, ..., U_n$ be independent, uniform random variables on the interval from zero to one. Find the density of the following random variable

$$\prod_{i=1}^{n} U_i$$

   Note that I am not asking about an asymptotic distribution. Find the density for an arbitrary, but finite, n.

3. Let X be a continuous variable with an invertible distribution function (cdf) F.

(a) Find the distribution of $F(X)$.

(b) Find the distribution of $2\min\{F(X), 1 - F(X)\}$.

4. Let Y_1, Y_2 be independent $N(0, 1)$ random variables. Find the moment generating function of $Y = Y_1 Y_2$.

5. Let $Y_1, ..., Y_n$ be a sequence of independent and identical shifted exponential densities. In other words, Y_i has density

$$f(y) = e^{-(y-\alpha)} I_{\{y \geq \alpha\}}$$

Prove that the first order statistic, $Y_{(1)}$, converges in probability to α as $n \to \infty$.

6. The exponential distribution is commonly parameterized in two different ways. One is by its mean, and the corresponding density function is

$$g(y) = \frac{1}{\theta} e^{-\frac{y}{\theta}} I\{y \geq 0\}$$

The other is by its rate, and the corresponding density function is

$$f(x) = \lambda e^{-\lambda x} I\{x \geq 0\}$$

(a) Suppose that $Y_1, Y_2, ..., Y_n$ are iid random variables with a common density $g(y)$ from above. The sample mean of the Y_i's seems like a natural estimator for θ in this case.

i. What does the sample mean converge to in probability as $n \to \infty$? Why?

ii. Because the underlying distribution is exponential, the exact distribution of the sample mean is Gamma. However, for sufficiently large n it could be approximated using another distribution. What is it? Why? Be precise(i.e. give the parameters of the approximate distribution).

(b) Suppose that $X_1, X_2, ..., X_n$ are iid random variables with a common density $f(x)$ from above.

i. Give a function of the sample mean which converges to λ as $n \to \infty$?

ii. For large n, give an approximate distribution for this function of the sample mean.