STAT 513
Homework 1

Due 9.12.2012

1. Verify the following relations.
   (a) $(A \cup B)^c = A^c \cap B^c$.
   (b) $(A \cup B) - B = A - (A \cap B) = A \cap B^c$.
   (c) $A \cap A = A \cup A = A$.
   (d) $(A - (A \cap B)) \cup B = A \cup B$.
   (e) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.
2. Find simple expressions for
   (a) $(A \cup B) \cap (A \cup B^c)$.
   (b) $(A \cup B) \cap (A \cup B^c) \cap (A^c \cup B)$.
   (c) $(A \cup B) \cap (B \cup C)$.
3. State which of the following relations are correct and which incorrect.
   (a) $(A \cup B) - C = A \cup (B - C)$.
   (b) $A \cap B \cap C = A \cap B \cap (C \cup B)$.
   (c) $A \cup B = (A - (A \cap B)) \cup B$.
   (d) $(A \cup B) - A = B$.
   (e) $(A \cup B \cup C)^c = A^c \cap B^c \cap C^c$.
4. Let A, B, C be three arbitrary events. Find expressions for the events that of A, B, C:
   (a) Only A occurs.
   (b) Both A and B, but not C, occur.

(c) All three events occur.

(d) At least one occurs.

(e) At least two occur.

(f) One and no more occurs.

(g) Two and no more occur.

(h) None occurs.

(i) No more than two occur.

5. Assume A, B, and C are disjoint events and that $P(A) = 0.60$, $P(B) = 0.30$, and $P(C) = 0.10$. Find the probabilities of all the other associated events in the smallest possible sigma-field, $\mathcal{F}$.

6. Show
$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$
Suggest and prove the appropriate generalization for a union of n sets using induction.

7. Three players A, B, and C, take turns at a game according to the following rules. At the start, A and B play while C is out. The loser is replaced by C and at the second trial the winner of the first trial plays C, while the loser is out. The game continues in this way unless a player wins twice in succession, thus becoming the winner of the game. Ignoring the possibility of ties at individual trials and assuming the players are of equal skill, show that

   (a) the probability that A wins is $5/14$,

   (b) the probability that C wins is $2/7$, and

   (c) the probability that no decision is reached at or before the kth turn is $2^{-(k-1)}$.

8. (a) Show that $|P(A) - P(B)| \leq P(A \Delta B)$.

   (b) Establish the triangle inequality:
$$P(A \Delta C) \leq P(A \Delta B) + P(B \Delta C)$$

9. (a) Find a simple example of three events A, B, and C which are pairwise independent but not mutually independent.

(b) Find a simple example of three events A, B, and C for which $P(A \cap B \cap C) = P(A)\ P(B)\ P(C)$ but such that the conditions $P(A \cap B) = P(A)\ P(B)$, $P(B \cap C) = P(B)\ P(C)$, and $P(B \cap C) = P(B)\ P(C)$ all fail.

10. Use the definitions given in class for $limsup_{n\to\infty} A_n$ and $liminf_{n\to\infty} A_n$.

    (a) Show these are equal whenever A_n is monotnone and equal in this case to the original definition given in class for $lim_{n\to} A_n$.

    (b) Find an example for which the $limsup$ and the $liminf$ are unequal.

    (c) Find an example for which A_n is not monotone, but for which the $limsup$ and $liminf$ are equal.