STAT 513
Homework 3

1. Prove the following:

   (a) $\lim_{x\to-\infty} F(x,y) = \lim_{y\to-\infty} F(x,y) = \lim_{x\to-\infty,\ y\to-\infty} F(x,y) = 0$.

   (b) $F(x,y)$ is non-decreasing in each of its variables and right continuous in each variable.

2. Suppose that X has a geometric distribution, i.e. $f(x) = P(X = x) = q^{x-1}p$ for $x = 1, 2, ...$ where $0 < p < 1$ and $q = 1 - p$. This is used to model the number of independent trials required to obtain a success with success probability q. Suppose that Y is another geometrically distributed random variable, which is independent of X. Let $Z = \max(X, Y)$.

   (a) Show that $P(X > x) = q^x$.

   (b) Show the memoryless property for X. Namely, show that $P(X > a + b|X > b) = P(X > a)$.

   (c) Find the probability mass function of Z.

   (d) Find the joint mass probability function of X and Z, i.e. $P(X = x, Z = z)$.

   (e) Find the conditional probability function of X given $Z = z$, i.e., compute $P(X = x|Z = z)$ for all $x, z = 0, 1, 2, 3, ...$

   (f) Find the conditional probability function of Z given $X = x$, i.e., compute $P(Z = z|X = x)$ for all $x, z = 0, 1, 2, 3, ...$

3. Suppose that X has pdf $f_X(x) = \lambda e^{-\lambda x} I_{\{0<x<\infty\}}$. (This type or random variable is said to have anexponential distribution with rate λ.) Show that X has the memoryless property, i.e. $P(X > x + y|X > x) = P(X > y)$ for positive real numbers x and y.

4. The random variables U_1 and U_2 are independent uniformly distributed random variables on $[0, 1]$, and R is the Euclidean distance from the origin $(0, 0)$ to (U_1, U_2). Find the pdf of R.

5. The probability that a radioactive substance gives of n β-particles in a unit of time is Poisson with parameter λ for $n = 0, 1, 2, ...$ The probability that a given particle will strike a counter and be registered is p.

   (a) Show the probability of registering n β-particles in the unit of time is also Poisson, and find the parameter. Show the number of registered β-particles is independent of the number of unregistered β-particles in the unit of time.

   (b) The substance is observed for two units of time during which 5 particles are given off. Find the probability that exactly 3 of these 5 particles are registered during the first of the two units of time.

6. Let X be a random variable with a finite variance and $Y = min(X, c)$ for some constant c. Show that the variance of Y exists and is less than or equal to the variance of X. (Hint: Show that without loss of generality, one may take $c = 0$. Then examine $cov(Y, Z)$, where $Z = max(X, c)$. Express X in terms of Y and Z.

7. X is an absolutely continuous random variable which is independent of the random variable Y. Show $X + Y$ is absolutely continuous by exhibiting a pdf for $X + Y$. Note that it is not assumed that Y is absolutely continuous.

8. Suppose X has a standard Cauchy distribution and $Y = \log(|X|)$.

   (a) Derive explicitly the pdf of Y.

   (b) Show that the interquartile range of Y is $2\log(1 + \sqrt{2})$.

   (c) Find the moment generating function of Y. What is its domain of definition? An interesting fact from calculus is that

   $$\int_0^\infty \frac{x^{a-1}}{1+x} dx = \pi cosec(\pi a),\ 0 < a < 1.$$

   (d) Find the mean and standard deviation of Y.