STAT 513
Homework 4

Due 11.2.2012

1. Assume that X_1 and X_2 are standard normal and independent. Show that X_1/X_2 is Cauchy. Do this directly using the CDF method. Show also that
$$\frac{X_1}{X_1 + X_2}$$
is Cauchy distributed.

2. Let $X_1 \sim B_1(\mu, \nu)$ and $X_2 \sim E(\lambda, \mu + \nu)$ be independent random variables with μ, ν, and λ all greater than zero. Find a pdf for $X_1 X_2$ and identify its distribution.

3. Let $X_1 \sim B_1(\mu, \nu)$ and $X_2 \sim B_1(\mu + \nu, \lambda)$ be independent random variables with μ, ν, and λ all greater than zero. Find a pdf for $X_1 X_2$ and identify its distribution.

4. Suppose $X_1 \sim E(\lambda, \nu)$ and $X_2 \sim E(\lambda, \nu + 1/2)$ are independent. Show that $Y = 2\sqrt{X_1 X_2} \sim E(\lambda, 2\nu)$. The following facts may be helpful (and you do not need to prove):
   (a) $\int_0^\infty y^{-1/2} e^{-(y + x/y)} dy = \sqrt{\pi} e^{-2\sqrt{x}}$
   (b) $\sqrt{\pi}\Gamma(2\nu) = 2^{2\nu - 1}\Gamma(\nu)\Gamma(\nu + 1/2)$

5. (a) Find the joint probability distribution function of
$$X = \frac{\sin(U_1)}{\sin(U_2)} \quad Y = \frac{\cos(U_1)}{\cos(U_2)}$$
   where $U_1 \sim Unif(0, 2\pi)$ and $U_2 \sim Unif(0, 2\pi)$.
   (b) Find the pdf of X.

6. Let $(X, Y)' \sim N((0,0)', \Sigma)$ where Σ is a 2×2 symmetric matrix with both diagonal entries equal to one and the off diagonal entries equal to ρ which is a value between -1 and 1. Find $P(X \geq 0, Y \geq 0)$ as a function of ρ. To do this multiply $(X, Y)'$ by an appropriate orthogonal matrix to produce two independent random variables. Then rescale them to make them standard normal. Express the event in question in terms of these rescaled random variables.

7. A needle of unit length is thrown at a large table marked with parallel horizontal and vertical lines at a fixed distance 2 from one another. Show that the probability the needle intersects

   (a) a horizontal line is $\frac{1}{\pi}$ (likewise for a vertical line),

   (b) a horizontal and a vertical line is $\frac{1}{4\pi}$

   Assume the midpoint of the needle and the angle formed by the needle and one set of parallel lines are independent random variables each uniformly distributed over its range.

8. Two persons, Bob and Roger, agree to meet at a certain place some time between noon and one o'clock. Each will wait 15 minutes for the other to arrive. Assume their arrival times are independent and uniformly distributed.

   (a) Find the probability that Bob and Roger meet.

   (b) Assuming neither Bob nor Roger wait past 1 o'clock, describe the joint distribution of the time each waits for the other and compute the correlation coefficient of their waiting times.

9. Let U_1 and U_2 be independent standard uniform random variables. Show that

   $$X_1 = \cos(2\pi U_1)\sqrt{-2\log(U_2)}$$

   and

   $$X_2 = \sin(2\pi U_1)\sqrt{-2\log(U_2)}$$

   are independent standard normal random variables.