STAT 513
Homework 5

Due 11.16.2012

1. For any two random variables X and Y with finite variances, prove that
   (a) $Cov(X,Y) = Cov(X, E[Y|X])$
   (b) X and $Y - E[Y|X]$ are uncorrelated
   (c) $Var[Y - E[Y|X]] = E[Var[Y|X]]$
   (d) If $E[Y|X] = 1$, then $Var[XY] \geq Var[X]$.

2. Two players take turns shooting at a target, with each shot by player i hitting the target with probability $p_i, i = 1, 2$. Shooting ends when two consecutive shots hit the target. Let μ_i denote the mean number os shots taken when player i shoots first, $i = 1, 2$.
   (a) Find μ_i and μ_2
   (b) Let h_i denote the mean number of times that the target is hit when player i shoots first, $i = 1, 2$. Find h_1 and h_2.

3. Explicitly verify the formula for the characteristic function of a Gamma distribution is
$$\phi(t) = \left(\frac{1}{1 - i\lambda t}\right)^{\nu}$$

4. Suppose that the sequence of random variables $X_1, X_2, ...$ are independent and identically distributed. Further define $Y = X_1 + X_2 + ... + X_N$ where N is a Poisson random variable with mean λ, which is independent of the sequence of random variables.
   (a) Find an expression for the characteristic function of Y in terms of the characteristic function for the common distribution of the X_i.

(b) Assuming that the first and second moments of each X_i are finite, find expressions for the mean and variance of Y. (Are any additional assumptions necessary?)

5. Find the characteristic function if a random variable has density

$$f(x) = (1 - |x|)I_{\{-1<x<1\}}$$

6. Assume $X \sim N(\mu_X, \sigma^2)$ and $Y \sim N(\mu_Y, \sigma^2)$ and independent. Find the distribution for $U = X + Y$ and $V = X - Y$.

7. The conditional expectation, $E[Y|X]$, is a function of X. The expression

$$E[Y - g(X)]^2$$

is minimized when $g(X) = E[Y|X]$. Show this fact.

8. Assume $(X, Y)'$ is bivariate normal with parameters $\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho$. Verify that

(a) That the marginal distribution of X is $N(\mu_X, \sigma_X^2)$.

(b) That the correlation between X and Y is ρ. (You may want to use conditioning here.)

(c) That the conditional density of Y given $X = x$ is $N(\mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(x - \mu_X), \sigma_Y^2(1 - \rho^2))$

9. Let X be multivariate normal with mean vector $\mu' = (2, -3, 1)$ and covariance matrix

$$\Sigma = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 2 & 2 \end{pmatrix}$$

(a) Find the distribution of $3X_1 - 2X_2 + X_3$.

(b) Find a vector a such that X_2 and

$$X_2 - a' \begin{pmatrix} X_1 \\ X_3 \end{pmatrix}$$

are independent