

STAT 513

Homework 6

Due 12.10.2012

1. Show the following:
 - (a) If $X_n \rightarrow X$ in law and $Y_n \rightarrow 0$ in probability, then $X_n + Y_n \rightarrow X$ in law.
 - (b) If $X_n \rightarrow X$ in probability and $Y_n \rightarrow Y$ in probability, then $X_n + Y_n \rightarrow X + Y$ in probability.
 - (c) If $X_n \rightarrow X$ in probability and $a_n \rightarrow a$, then $a_n X_n \rightarrow aX$ in probability.
2. If the RVs X_n have means $\mu_n = \mu + 1/n$ are independent and exponentially distributed with $\mu > 0$, show the weak law of large numbers applies, i.e.

$$\frac{X_1 + \dots + X_n}{n} \rightarrow \mu$$

in probability.

3. If the RVs $X_n, n = 1, 2, \dots$ are independent and each is Cauchy with centrality parameter m and shape parameter b , show that

$$\frac{X_1 + \dots + X_n}{n}$$

also has the Cauchy distribution for all $n = 1, 2, \dots$

4. If each X_n is Poisson with mean λ_n , where $\lambda_n \rightarrow \infty$, show, for all $a < b$, that

$$P\left(a < \frac{X_n - \lambda_n}{\sqrt{\lambda_n}} < b\right) \rightarrow \Phi(b) - \Phi(a)$$

5. Use the Central Limit Theorem to show the following (purely analytic) results:

- (a) $\lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{n^k}{k!} e^{-n} = 1/2$
- (b) $\lim_{n \rightarrow \infty} \int_0^n t^{n-1} e^{-t} dt / \Gamma(n) = 1/2$
- (c) $\lim_{n \rightarrow \infty} \sum_{k=0}^{x\sqrt{n}+n} \frac{n^k}{k!} e^{-n} = \Phi(x)$
- (d) $\lim_{n \rightarrow \infty} \int_0^{x\sqrt{n}+n} t^{n-1} e^{-t} dt / \Gamma(n) = \Phi(x)$
6. Let X_1, X_2, \dots be a sequence of independent RVs with common distribution, finite variance σ^2 , and mean μ . Let N be independent of this sequence and have a geometric distribution given by $P(N = n) = qp^n$, $n \geq 0$, where $q = 1 - p$, $0 < p < 1$. Define the sum $S_n = X_1 + \dots + X_n$.
- (a) Determine the limiting distribution of qN as $q \rightarrow 0$.
- (b) Find the characteristic function of S_N .
- (c) If $\mu \neq 0$, show qS_N/μ has a limiting distribution as $q \rightarrow 0$, and identify it.
- (d) If $\mu = 0$, show $\sqrt{2q}S_N/\sigma$ has a limiting distribution as $q \rightarrow 0$, and identify it.
7. Calculate $P(|X - \mu| \geq k\sigma)$ for $X \sim Unif(0, 1)$ and $X \sim Exp(\lambda)$ and compare your answers to the bound from Chebychev's inequality.
8. If X is a random variable whose MGF exists, prove that $P(X \geq 0) \leq Ee^{tX}$ for all $t \geq 0$ for which the mgf is defined.
9. A random variable X is defined by $Z = \log X$, where $EZ = 0$. Is EX greater than, less than, or equal to 1?
10. Let X be a binomial random variable with parameters n and p show that for $i > np$,
- (a) the minimum of $e^{-it} E[e^{tX}]$ occurs when t satisfies

$$e^t = \frac{iq}{(n-i)p}$$

- (b) and

$$P(X \geq i) \leq \frac{n^n}{i^i (n-i)^{n-i}} p^i (1-p)^{n-i}$$

11. If X is a Poisson random variable with mean λ show that for $i < \lambda$,

$$P(X \leq i) \leq \frac{e^{-\lambda}(e\lambda)^i}{i^i}$$

12. Let X be a non-negative absolutely continuous random variable having a non-increasing density function Show that

$$f(x) \leq \frac{2EX}{x^2}, \quad x > 0$$

13. If X has mean μ and standard deviation σ , the ratio $r = |\mu|/\sigma$ is called the measurement signal to noise ratio of X . Similarly, we define the relative deviation of X from its signal (μ) as

$$D = \left| \frac{X - \mu}{\mu} \right|$$

Prove that

$$P(D \leq \alpha) \geq 1 - \frac{1}{r^2\alpha^2}$$