Statistics 416 Name: ______________________

First Midterm Exam

Make sure to put all your answers in the space provided. Also, be sure to give complete answers and to show your work. In other words, you need to not only answer the questions, but also to convince me of your answer. It should go without saying that this should be your own work.

---

1. The Markov chain associated with a manufacturing process may be described as follows: A part to be manufactured will begin the process by entering step 1. After step 1, 20% of the parts must be reworked, i.e. returned to step 1, 10% of the parts are thrown away, and 70% proceed to step 2. After step 2, 5% of the parts must be returned to step 1, 10% to step 2, 5% are scrapped, and 80% emerge to be sold for a profit.

   (a) (10 pts) Formulate a Markov chain based on this information.

$$\begin{array}{ll} 1 & \text{Step1} \rightarrow \\ 2 & \text{step2} \rightarrow \\ 3 & \text{scrap} \rightarrow \\ 4 & \text{profit} \rightarrow \end{array} \begin{pmatrix} 0.2 & 0.7 & 0.1 & 0 \\ 0.05 & 0.1 & 0.05 & 0.8 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

   (b) (15 pts) Compute the probability that a part is scrapped in the production process.

$$u_i = \text{Prob}(\text{absorbed in state } 3 \mid X_0 = i)$$

$$\begin{cases} u_1 = 0.2u_1 + 0.7u_2 + 0.1\cancel{u_3}^{\to 1} + 0\cancel{u_4}^{\to 0} \\ u_2 = 0.05u_1 + 0.1u_2 + 0.05\cancel{u_3}^{\to 1} + 0.8\cancel{u_4}^{\to 0} \end{cases}$$

$$\rightarrow \begin{cases} 0.8u_1 = 0.7u_2 + 0.1 \\ 0.9u_2 = 0.05u_1 + 0.05 \end{cases}$$

$$\rightarrow \begin{cases} u_1 = \frac{7}{8}u_2 + \frac{1}{8} \\ u_2 = \frac{5}{90}u_1 + \frac{5}{90} \end{cases}$$

$$\rightarrow u_1 = \frac{35}{720}u_1 + \frac{35}{720} + \frac{90}{720} \Rightarrow u_1 = \frac{125}{685} = \frac{25}{137}$$

u_1 is the final answer since the process starts in state 1

2. There are two light bulbs in a garage. When both are burned out they are replaced, and the next day starts with two working light bulbs. Suppose that when both are working, one of the two will go out with probability 0.02. (Each has probability 0.01, but we ignore the possibility of losing two on the same day.) However, when only one bulb is there, it will burn out with probability 0.05.

(a) (15 pts) What is the long-run fraction of time that there is exactly one bulb working?

$$\begin{array}{lll} 0 & \text{2 bulbs work} & \\ 1 & \text{1 bulb works} & \\ 2 & \text{0 bulbs work} & \end{array} \begin{pmatrix} 0.98 & 0.02 & 0 \\ 0 & 0.95 & 0.05 \\ 1 & 0 & 0 \end{pmatrix}$$

$\pi_0 = 0.98\pi_0 + \pi_2$

$\pi_1 = 0.02\pi_0 + 0.95\pi_1$

$\pi_2 = 0.05\pi_1$

$1 = \pi_1 + \pi_2 + \pi_3$

$\pi_0 = \frac{5}{2}\pi_1$

$1 - \pi_0 - \pi_1 = 0.05\pi_1$

$1 - \frac{7}{2}\pi_1 = \frac{1}{20}\pi_1$

$1 = \frac{71}{20}\pi_1$

$\pi_1 = \frac{20}{71}$

Final Answer Long-Run Fraction of time.

(b) (10 pts) What is the expected time between light bulb replacements?

Set state 2 to absorbing.

$$\begin{pmatrix} 0.98 & 0.02 & 0 \\ 0 & 0.95 & 0.05 \\ 0 & 0 & 1 \end{pmatrix}$$

$V_i = E[T \mid X_0 = i]$

V_0 will be the answer (or $V_0 + 1$)

$V_0 = 1 + 0.98V_0 + 0.02V_1$

$V_1 = 1 + 0.95V_1$

$V_1 = 20$

$0.02V_0 = 1 + 0.4$

$\boxed{V_0 = 70}$

3. A bacterium either dies, survives (but does not procreate), or splits from one generation to the next. If we start with one bacterium, there will be zero, one, or two in the next generation. The probability of dying is twice as high as surviving (but not procreating), and the probability of surviving is twice as high as splitting. Given that there is one bacterium at generation zero.

   (a) (10 pts) Find the generating function for the number of offspring for one bacterium.

$$\phi(s) = \frac{4}{7} + \frac{2}{7}s + \frac{1}{7}s^2$$

$P(\xi = 0) = 4p$

$P(\xi = 1) = 2p$

$P(\xi = 2) = p$

Sum is $1 \Rightarrow p = \frac{1}{7}$

   (b) (10 pts) What is the probability of extinction at the second generation?

$$u_0 = P(X_0 = 0) = 0$$

$$u_1 = P(X_1 = 0) = \phi(u_0) = \frac{4}{7}$$

$$u_2 = P(X_2 = 0) = \phi(u_1) = \frac{4}{7} + \frac{2}{7}\,\frac{4}{7} + \frac{1}{7}\left(\frac{4}{7}\right)^2$$

$$= \frac{196 + 56 + 16}{7^3}$$

$$= \frac{270}{7^3}$$

4. Suppose that a Markov chain with states $\{0, 1, 2\}$ has the following probability transition matrix.

$$P = \begin{pmatrix} 0 & 3/4 & 1/4 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}$$

and the initial conditions for the Markov chain are given via a row vector as

$$(1/4, 1/2, 1/4)$$

answer to a)

(a) (10 pts) Find $P(X_2 = 2 | X_0 = 0)$.

$$P^2 = \begin{pmatrix} 0 & 3/4 & 1/4 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 3/4 & 1/4 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{8} & \boxed{\frac{3}{8}} \\ \frac{1}{4} & \frac{5}{8} & \frac{1}{8} \\ \frac{1}{4} & \frac{3}{8} & \frac{3}{8} \end{pmatrix}$$

(b) (10 pts) Find $P(X_2 = 2, X_0 = 0)$.

$$P(X_2=2, X_0=0) = P(X_2=2|X_0=0)P(X_0=0)$$
$$= \frac{3}{8} \cdot \frac{1}{4} = \frac{3}{32}$$

(c) (10 pts) Find $P(X_2 = 2)$.

$$P(X_2=2) = \sum_{i=0}^{2} P(X_2=2 | X_0=i) P(X_0=i)$$
$$= \frac{3}{8} \cdot \frac{1}{4} + \frac{1}{8} \cdot \frac{1}{2} + \frac{3}{8} \frac{1}{4}$$
$$= \frac{3}{32} + \frac{1}{16} + \frac{3}{32}$$
$$= \frac{8}{32} = \frac{1}{4}$$