MATH/STAT 416 Second Midterm Exam
April 11, 2012.

- Make sure to put all your answers in the space provided.
- Be sure to give complete justification for your answers and to show your work. Your goal is to give a convincing argument to me of the answer that you provide.
- No calculators or other electronic enhancements are permitted.
- You are not to receive nor GIVE aid to other students on the exam.
- Write your name below if you understand these instructions.

Name: ______________________________

1. (60 pts) Suppose that a hospital is carrying out a clinical trial for a treatment technique for skin cancer. Patients with skin cancer arrive to the hospital at a rate of 5 per day. Sixty percent of these patients are female, and forty percent are male.

   (a) Describe a continuous time stochastic model (from those we have discussed in the course) to represent the situation described. What assumptions are required to use the model you have described?

$N(t)$ represents the total # of patients by time t. We assume it is a Poisson process with rate 5.

\# of men
$$\hookrightarrow X_M(t) = \sum_{i=1}^{N(t)} \xi_i$$

ξ_i are iid Bernoulli with $p = 0.4$

\# of women
$$\hookrightarrow X_W(t) = N(t) - \sum_{i=1}^{N(t)} \xi_i$$

$X_M(t)$ is therefore Poisson with rate 2 and is independent of $X_W(t)$, which is also Poisson but with rate 3.

   (b) Assume that you want 10 of each type of patient in order to have balance in the trial. What is the mean time and the variance of the time until 10 men are selected for the study?

S_{10}^{M} is arrival time of the 10th man and is thus Gamma w/ mean $\frac{n}{\lambda} = \frac{10}{2} = 5$ and variance $\frac{n}{\lambda^2} = \frac{10}{4} = 2.5$.

(c) What is the variance of the number of women selected by the time of the selection of the tenth man? $\lambda_w = 3$

$X_w(S_{10}^m)$ is the # of women at the time of the 10th man.

$$Var(X_w(S_{10}^m)) = E[Var(X_w(S_{10}^m)|S_{10}^m)] + Var[E(X_w(S_{10}^m)|S_{10}^m)]$$
$$= E[3S_{10}^m] + Var[3S_{10}^m]$$
$$= 3 \cdot ES_{10}^m + 9\,Var\,S_{10}^m = 3 \cdot 5 + 9 \cdot \frac{5}{2} = \frac{30}{2} + \frac{45}{2}$$
$$= \frac{75}{2}$$

(d) If 10 men are selected within 4 days, what is the probability that exactly 10 women are selected in the same four days?

$$P(X_w(4) = 10 \mid X_m(4) = 10)$$
$$= P(X_w(4) = 10) \quad (X_w(t) \text{ is independent of } X_m(t))$$
$$= \frac{(3 \cdot 4)^{10} e^{-12}}{10!}$$
$$= \frac{12^{10} e^{-12}}{10!}$$

(e) Suppose that only one man with skin cancer arrives in the first day. Assuming the study started at midnight, was he more likely to arrive before noon or after noon? Explain your answer.

No, arriving before noon is as likely as arriving after noon.

Conditioned on one man arriving in the first day, the time within that day ~~of~~ arrival is uniform from midnight to midnight. The ~~mean~~ median is thus noon with 1/2 prob ~~of~~ being before and $\frac{1}{2}$ after.

(f) If we continue to suppose only one man arrived in the first day, what is the expected number of patients with skin cancer to arrive in that first day?

$$E[X_m(1) + X_w(1) \mid X_m(1) = 1]$$

$$= E[X_m(1) \mid X_m(1) = 1] + E[X_w(1) \mid X_m(1) = 1]$$

independence

$$= 1 + 3 \cdot 1$$

$$= 4$$

2. (40 pts) Two people are working in a small office selling shares in a mutual fund. Each is either on the phone or not. Suppose salesman one is on the phone for an exponentially distributed amount of time with rate $\mu_1 = 2$. Salesman two is on the phone for an exponentially distributed amount of time with rate $\mu_2 = 3$. The rate for an individual salesman to stay off the phone is $\lambda = 3$.

   (a) Write down the Q matrix for a continuous time Markov chain representing WHICH salesman is on the phone. (The state space coould be described with $\{0, 1, 2, 12\}$, for example.)

$$
Q \quad \begin{array}{c|cccc}
 & 0 & 1 & 2 & 12 \\
\hline
0 & -2\lambda = -6 & \lambda = 3 & \lambda = 3 & 0 \\
1 & \mu_1 = 2 & -\mu_1 - \lambda = -2 - 3 & 0 & \lambda = 3 \\
2 & \mu_2 = 3 & 0 & -\mu_2 - \lambda = -6 & \lambda = 3 \\
12 & 0 & \mu_2 = 3 & \mu_1 = 2 & -\mu_1 - \mu_2 = -5
\end{array}
$$

   (b) Can one easily formulate a Markov chain, $X(t)$, which represents only the number of salesmen on the phone? If so, write down the new Q matrix for this modified chain. If not, then why not?

No, if one knew that there was one person on the phone, what would be the rate to move to zero? We need to know which one was on the phone.

(c) Find the stationary distribution for the chain in part (a).

See other sheet

(d) Find the v_i and P^*_{ij} for the continuous time Markov chain in (a).

$v_0 = 6$

$v_1 = 5$

$v_2 = 6$

$v_{12} = 5$

$$P^* = \begin{pmatrix} & 0.5 & 0.5 & 0 \\ 0.4 & & 0 & 0.6 \\ 0.5 & 0 & & 0.5 \\ 0 & 0.6 & 0.4 & \end{pmatrix}$$

$$-6\pi_1 + 2\pi_2 + 3\pi_3 = 0$$
$$3\pi_1 - 5\pi_2 + 3\pi_4 = 0$$
$$3\pi_1 - 6\pi_3 + 2\pi_4 = 0$$
$$3\pi_2 + 3\pi_3 - 5\pi_4 = 0$$

Take first 3 eqns plus $\pi_4 = 1 - \pi_2 - \pi_3 - \pi_4$

$$-6\pi_1 + 2\pi_2 + 3\pi_3 = 0 \qquad *$$
$$-8\pi_2 - 3\pi_3 = -3$$
$$\pi_1 - 2\pi_2 - 8\pi_3 = -2 \qquad **$$

$$-(* + 6**) \rightarrow \quad 10\pi_2 + 45\pi_3 = 12$$
$$8\pi_2 + 3\pi_3 = 3$$

$$-110\pi_2 = -33$$
$$\pi_2 = 0.3$$
$$\pi_3 = 1 - \frac{8}{3}\pi_2 = 0.2$$
$$(\text{from } *) \quad \pi_1 = \frac{1}{3}\pi_2 + \frac{1}{2}\pi_3 = 0.2$$
$$(\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1) \quad \pi_4 = 0.3$$