# STAT 513
# Homework 1

Due 9.6.2011

1. Verify the following relations.
   (a) $(A \cup B)^c = A^c \cap B^c$.
   (b) $(A \cup B) - B = A - (A \cap B) = A \cap B^c$.
   (c) $A \cap A = A \cup A = A$.
   (d) $(A - (A \cap B)) \cup B = A \cup B$.
   (e) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.
2. Find simple expressions for
   (a) $(A \cup B) \cap (A \cup B^c)$.
   (b) $(A \cup B) \cap (A \cup B^c) \cap (A^c \cup B)$.
   (c) $(A \cup B) \cap (B \cup C)$.
3. State which of the following relations are correct and which incorrect.
   (a) $(A \cup B) - C = A \cup (B - C)$.
   (b) $A \cap B \cap C = A \cap B \cap (C \cup B)$.
   (c) $A \cup B = (A - (A \cap B)) \cup B$.
   (d) $(A \cup B) - A = B$.
   (e) $(A \cup B \cup C)^c = A^c \cap B^c \cap C^c$.
4. Let A, B, C be three arbitrary events. Find expressions for the events that of A, B, C:
   (a) Only A occurs.
   (b) Both A and B, but not C, occur.

(c) All three events occur.

(d) At least one occurs.

(e) At least two occur.

(f) One and no more occurs.

(g) Two and no more occur.

(h) None occurs.

(i) No more than two occur.

5. Assume A, B, and C are disjoint events and that $P(A) = 0.60$, $P(B) = 0.30$, and $P(C) = 0.10$. Find the probabilities of all the other associated events in the smallest possible sigma-field, $\mathcal{F}$.

6. Show
$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$
Suggest and prove the appropriate generalization for a union of $n$ sets using induction.

7. Three players $A$, $B$, and $C$, take turns at a game according to the following rules. At the start, $A$ and $B$ play while $C$ is out. The loser is replaced by $C$ and at the second trial the winner of the first trial plays C, while the loser is out. The game continues in this way unless a player wins twice in succession, thus becoming the winner of the game. Ignoring the possibility of ties at individual trials and assuming the players are of equal skill, show that

   (a) the probability that $A$ wins is $5/14$,

   (b) the probability that $C$ wins is $2/7$, and

   (c) the probability that no decision is reached at or before the $k$th turn is $2^{-(k-1)}$.

8. Define $A\Delta B = (A - B) \cup (B - A)$.

   (a) Show that $|P(A) - P(B)| \leq P(A\Delta B)$.

   (b) Establish the triangle inequality:
$$P(A\Delta C) \leq P(A\Delta B) + P(B\Delta C)$$

9. (a) Find a simple example of three events A, B, and C which are pairwise independent but not mutually independent.

(b) Find a simple example of three events $A$, $B$, and $C$ for which $P(A \cap B \cap C) = P(A)\ P(B)\ P(C)$ but such that the conditions $P(A \cap B) = P(A)\ P(B)$, $P(B \cap C) = P(B)\ P(C)$, and $P(A \cap C) = P(A)\ P(C)$ all fail.

10. Use the definitions given in class for $limsup_{n\to\infty} A_n$ and $liminf_{n\to\infty} A_n$.

    (a) Show these are equal whenever $A_n$ is monotnone and equal in this case to the original definition given in class for $lim_{n\to} A_n$.

    (b) Find an example for which the $limsup$ and the $liminf$ are unequal.

    (c) Find an example for which $A_n$ is not monotone, but for which the $limsup$ and $liminf$ are equal.

11. Two dice are thrown. Let $A$ be the event that the sum of the faces is odd, and $B$ be the even of at least one ace (i.e. a one comes up). Describe the events $A \cap B$, $A \cup B$, and $A \cap B^c$. Find their probabilities assuming that all 36 sample points have equal probability.

12. A coin is tossed until for the first time the same result appears twice in succession. To every possible outcome requiring $n$ tosses attribute probability $1/(2^{n-1})$. Describe the sample space. Find the probability of the following events:

    (a) the experiment ends before the sixth toss

    (b) an even number of tosses is required

13. (a) Find the probability that among three random digits there appear exactly 1, 2, or 3 different ones.

    (b) Do the same for four random digits.

14. If $n$ balls are placed at random into n cells, find the probability that exactly one cell remains empty.

15. $A$ throws six dice and wins if he scores at least one ace. $B$ throws twelve dice and wins if he scores at least two aces. Who has the greater probability to win?