STAT 513
Homework 3

1. Prove the following:

   (a) $\lim_{x \to -\infty} F(x, y) = \lim_{y \to -\infty} F(x, y) = \lim_{x \to -\infty,\ y \to -\infty} F(x, y) = 0$.

   (b) $F(x, y)$ is non-decreasing in each of its variables and right continuous in each variable.

2. Suppose that X has a geometric distribution, i.e. $f(x) = P(X = x) = q^{x-1}p$ for $x = 1, 2, ...$ where $0 < p < 1$ and $q = 1 - p$. This is used to model the number of independent trials required to obtain a success with success probability p. Suppose that Y is another geometrically distributed random variable with the same probability mass function, which is independent of X. Let $Z = \max(X, Y)$.

   (a) Show that $P(X > x) = q^x$.

   (b) Show the memoryless property for X. Namely, show that $P(X > a + b | X > b) = P(X > a)$.

   (c) Find the probability mass function of Z.

   (d) Find the joint mass probability function of X and Z, i.e. $P(X = x, Z = z)$.

   (e) Find the conditional probability function of X given $Z = z$, i.e., compute $P(X = x | Z = z)$ for all $x, z = 0, 1, 2, 3, ...$

   (f) Find the conditional probability function of Z given $X = x$, i.e., compute $P(Z = z | X = x)$ for all $x, z = 0, 1, 2, 3, ...$

3. The probability that a radioactive substance gives of n β-particles in a unit of time is Poisson with parameter λ for $n = 0, 1, 2, ...$ In other words,

$$P(X = n) = \frac{\lambda^n e^{-\lambda}}{n!}$$

The probability that a given particle will strike a counter and be registered is p.

(a) Show the probability of registering n β-particles in the unit of time is also Poisson, and find the parameter. Show the number of registered β-particles is independent of the number of unregistered β-particles in the unit of time.

(b) The substance is observed for two units of time during which 5 particles are given off. Find the probability that exactly 3 of these 5 particles are registered during the first of the two units of time.

4. Let X be a random variable with a finite variance and $Y = min(X, c)$ for some constant c. Show that the variance of Y exists and is less than or equal to the variance of X. (Hint: Show that without loss of generality, one may take $c = 0$. Then examine $cov(Y, Z)$, where $Z = max(X, c)$. Express X in terms of Y and Z.

5. X is an absolutely continuous random variable which is independent of the random variable Y. Show $X + Y$ is absolutely continuous by exhibiting a pdf for $X + Y$. Note that it is not assumed that Y is absolutely continuous.

6. Suppose X has a standard Cauchy distribution (i.e. has a density $f(x) = (\pi(1 + x^2))^{-1}$ and $Y = \log(|X|)$).

   (a) Derive explicitly the pdf of Y.

   (b) Find the moment generating function of Y. What is its domain of definition? An interesting fact from calculus is that

$$\int_0^\infty \frac{x^{a-1}}{1+x} dx = \pi cosec(\pi a),\ 0 < a < 1.$$

   (c) Find the mean and standard deviation of Y.

7. Let X and Y have densities f and g such that $f(x) \geq g(x)$ for $x < a$ and $f(x) \leq g(x)$ for $x > a$. Prove that $E[X] \leq E[Y]$.

8. First an integer K is chosen according to a Poisson distribution (see above), next a point is chosen at random in $[K, K + 1)$. The random variable X is the value of this chosen pint. Find its distribution function.

9. Assume $F(x)$ is a distribution function for X. Show that the function $G(\cdot)$ defined by:
$$G(x) = 1 - F(-x-)$$
is then the distribution $-X$. (Recall that $F(a-) = \lim_{x \nearrow a} F(x)$.)

10. The probability densities $f_n(\cdot)$ are defined and continuous on the interval $(0, 1)$. The limit $f(u) = \lim_{n\to\infty} f_n(u)$ exists for every u and is a continuous function of $(0, 1)$. Is $f(u)$ necessarily a probability density on $(0, 1)$?

11. A is chosen at random on the curve $x^2 + y^2 = a^2$ with $y \geq 0$ and $-a \leq x \leq a$. A_1 is the projection of A on the line $y = 0$. X is the length of the line segment A_1A. Find $F(x)$ for X.