STAT 513
Homework 5
Due 11.11.2013

1. Suppose that Y is a random variable such that
$$EY^k = \frac{1}{4} + 2^{k-1} \quad k = 1, 2, ...$$
Determine the distribution of Y.

2. The aim of this problem is to prove the double angle formula
$$sin(2t) = 2sin(t)cos(t)$$
Let X and Y be independent Random variables, where $X \sim Unif(-1, 1)$ and Y assumes the values $+1$ and -1 with probabilities $1/2$.
   (a) Show that $Z = X + Y$ has the uniform distribution from -2 to 2 by finding the distribution function of Z.
   (b) Translate this fact into a statement about the corresponding characteristic functions.
   (c) Use the previous part to show the double angle formula.

3. Let $X_1, X_2, ...$ be a sequence of independent and identically distributed random variables each with standard Laplace density
$$f(x) = \frac{1}{2}e^{-|x|}$$
Suppose we sample from these random variables until a negative observation appears. Let Y be the sum of the observations thus obtained including the negative one. Show that the density of Y is
$$\begin{aligned} g(y) &= \frac{2}{3}e^{y} \quad y < 0 \\ &= \frac{1}{6}e^{-y/2} \quad y > 0 \end{aligned}$$

4. Let X and Y be independent standard exponential random variables. Find the conditional distribution of X given that $X + Y = c$, where c is a positive constant.

5. Suppose that the joint density of X and Y is

$$f(x, y) = xe^{-x-xy} I_{\{x>0,y>0\}}$$

Find $E[Y|X = x]$ and $E[X|Y = y]$.

6. The life T (in hours) of the lightbulb in an overhead projector follows a exponential random variable with rate $\lambda = 10$. During a normal week it is used a random number of times (distributed Poisson with mean 12) for lectures each lasting exactly one hour each. Find the probability that a projector with a newly installed lightbulb functions throughout a normal week without replacing the lightbulb.

7. For a bivariate normal distribution with $\mu_X, \mu_Y, \sigma_X^2 > 0, \sigma_Y^2 > 0$ and $|\rho| < 1$,

   (a) Find the conditional density for $X|Y = y$.

   (b) Determine the distribution of

$$\frac{X^2 - 2\rho XY + Y^2}{1 - \rho^2}$$

   when $\mu_X = \mu_Y = 0$ and $\sigma_X^2 = \sigma_Y^2 = 1$.

8. Let Y_1, Y_2, Y_3 be independent standard normal random variables. Now, let

$$\begin{aligned} X_1 &= Y_1 - Y_3 \\ X_2 &= 2Y_1 + Y_2 - 2Y_3 \\ X_3 &= -2Y_1 + 3Y_3 \end{aligned}$$

Determine the conditional distribution of X_2 given that $X_1 + X_3 = x$.

9. Let X be a three dimensional normal with mean and covariance

$$\mu = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \Sigma = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

Set $Y_1 = X_1 + X_2 + X_3$ and $Y_2 = X_1 + X_3$. Determine the conditional distribution of Y_1 given that $Y_2 = 0$

10. Let X_1, X_2, X_3 have a joint moment generating function

$$\psi(t_1, t_2, t_3) = exp\{2t_1 - t_3 + t_1^2 + 2t_2^2 + 3t_3^2 + 2t_1t_2 - 2t_1t_3\}$$

Determine the distribution of $X_1 + X_3$ given $X_1 + X_2 = 1$