STAT 513
Homework 6

Due 12.9.2013

1. Show the following:
   (a) If $X_n \to X$ in law and $Y_n \to 0$ in probability, then $X_n + Y_n \to X$ in law.
   (b) If $X_n \to X$ in probability and $Y_n \to Y$ in probability, then $X_n + Y_n \to X + Y$ in probability.
   (c) If $X_n \to X$ in probability and $a_n \to a$, then $a_n X_n \to aX$ in probability.

   Let $X_1, X_2, ..., X_n$ be independent Poisson random variables with means $\lambda_1, \lambda_2, ..., \lambda_n$ with each $\lambda_i > c$ for some fixed $c > 0$.

   (a) Find the distribution (i.e. the pmf) of X_1 given $X_1 + ... + X_n$.
   (b) Demonstrate explicitly that

   $$\frac{Y_n - \Lambda_n}{\sqrt{\Lambda_n}}$$

   converges in distribution to some random variable W as $n \to \infty$ where $Y_n = X_1 + ... + X_n$ and $\Lambda_n = \lambda_1 + \lambda_2 + ... + \lambda_n$. What is the distribution of W?

2. If the RVs X_n have means $\mu_n = \mu + 1/n$ are independent and exponentially distributed with $\mu > 0$, show the weak law of large numbers applies, i.e.

   $$\frac{X_1 + ... + X_n}{n} \to \mu$$

   in probability.

3. If the RVs $X_n, n = 1, 2, ...$ are independent and each is Cauchy with centrality parameter m and shape parameter b, sow that

$$\frac{X_1 + ... + X_n}{n}$$

also has the Cauchy distribution for all $n = 1, 2, ...$

4. If each X_n is Poisson with mean λ_n, where $\lambda_n \to \infty$, show, for all $a < b$, that

$$P\left(a < \frac{X_n - \lambda_n}{\sqrt{\lambda_n}} < b\right) \to \Phi(b) - \Phi(a)$$

5. Use the Central Limit Theorem to show the following (purely analytic) results:

   (a) $\lim_{n\to\infty} \sum_{k=0}^{n} \frac{n^k}{k!} e^{-n} = 1/2$

   (b) $\lim_{n\to\infty} \int_0^n t^{n-1} e^{-t} dt/\Gamma(n) = 1/2$

   (c) $\lim_{n\to\infty} \sum_{k=0}^{x\sqrt{n}+n} \frac{n^k}{k!} e^{-n} = \Phi(x)$

   (d) $\lim_{n\to\infty} \int_0^{x\sqrt{n}+n} t^{n-1} e^{-t} dt/\Gamma(n) = \Phi(x)$

6. Let $X_1, X_2, ...$ be a sequence of independent RVs with common distribution, finite variance σ^2, and mean μ. Let N be independent of this sequence and have a geometric distribution given by $P(N = n) = qp^n,\ n \geq 0$, where $q = 1 - p,\ 0 < p < 1$. Define the sum $S_n = X_1 + ... + X_n$.

   (a) Determine the limiting distribution of qN as $q \to 0$.

   (b) Find the characterisitic function of S_N.

   (c) If $\mu \neq 0$, show qS_N/μ has a limiting distribution as $q \to 0$, and identify it.

   (d) If $\mu = 0$, show $\sqrt{2q}S_N/\sigma$ has a limiting distribution as $q \to 0$, and identify it.

7. Calculate $P(|X - \mu| \geq k\sigma)$ for $X \sim Unif(0, 1)$ and $X \sim Exp(\lambda)$ and compare your answers to the bound from Chebychev's inequality.

8. If X is a random variable whose MGF exists, prove that $P(X \geq 0) \leq Ee^{tX}$ for all $t \geq 0$ for which the mgf is defined.

9. A random variable X is defined by $Z = logX$, where $EZ = 0$. Is EX greater than, less than, or equal to 1?

10. Let X be a binomial random variable with parameters n and p show that for $i > np$,

   (a) the minimum of $e^{-it}E\left[e^{tX}\right]$ occurs when t satisfies

   $$e^t = \frac{iq}{(n-i)p}$$

   (b) and

   $$P(X \geq i) \leq \frac{n^n}{i^i(n-i)^{n-i}} p^i(1-p)^{n-i}$$

11. If X is a Poisson random variable with mean λ show that for $i < \lambda$,

$$P(X \leq i) \leq \frac{e^{-\lambda}(e\lambda)^i}{i^i}$$

12. Let X be a non-negative absolutely continuous random variable having a non-increasing density function Show that

$$f(x) \leq \frac{2EX}{x^2}, \quad x > 0$$

13. If X has mean μ and standard deviation σ, the ratio $r = |\mu|/\sigma$ is called the measurement signal to noise ratio of X. Similarily, we define the relative deviation of X from its signal (μ) as

$$D = \left|\frac{X-\mu}{\mu}\right|$$

Prove that

$$P(D \leq \alpha) \geq 1 - \frac{1}{r^2\alpha^2}$$

14. Let $U_1, ..., U_n$ be independent, uniform random variables on the interval from zero to one. Find the density of the following random variable

$$\prod_{i=1}^{n} U_i$$

   (a) Find the density for an arbitrary, but finite, n.

   (b) Transform and scale this random variable so that a meaningful limiting distribution is obtained.

15. Let $Y_1, ..., Y_n$ be a sequence of independent and identical shifted exponential densities. In other words, Y_i has density

$$f(y) = e^{-(y-\alpha)} I_{\{y \geq \alpha\}}$$

Prove that the first order statistic, $Y_{(1)}$, converges in probability to α as $n \to \infty$.

16. The exponential distribution is commonly parameterized in two different ways. One is by its mean, and the corresponding density function is

$$g(y) = \frac{1}{\theta} e^{-\frac{y}{\theta}} I\{y \geq 0\}$$

The other is by its rate, and the corresponding density function is

$$f(x) = \lambda e^{-\lambda x} I\{x \geq 0\}$$

(a) Suppose that $Y_1, Y_2, ..., Y_n$ are iid random variables with a common density $g(y)$ from above. The sample mean of the Y_i's seems like a natural estimator for θ in this case.

i. (5 pts) What does the sample mean converge to in probability as $n \to \infty$? Why?

ii. (5 pts) Because the underlying distribution is exponential, the exact distribution of the sample mean is Gamma. However, for sufficiently large n it could be approximated using another distribution. What is it? Why? Be precise(i.e. give the parameters of the approximate distribution).

(b) Suppose that $X_1, X_2, ..., X_n$ are iid random variables with a common density $f(x)$ from above.

i. Give a function of the sample mean which converges to λ as $n \to \infty$?

ii. For large n, give an approximate distribution for this function of the sample mean.