Math/Stat 414.1 Exam 1 Name: ____________________

Instructions: Please read!

1. Do all work on this exam packet. It is okay to leave your answer unsimplified, as in $0.56\frac{15!}{6!9!}$ or $14e - 20$. This means that no calculator is needed.
2. Show all work for full credit! Small mistakes in arithmetic wont reduce credit if you show work; conversely, even a correct answer could get no credit without supporting work.
3. I will award partial credit where appropriate.

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1. Drawer A contains five pennies and three dimes, while drawer B contains three pennies and seven dimes. A drawer is selected in the following way. A die is rolled. If a 1 or 2 comes up, then drawer A is selected and a coin is drawn. If a 3,4,5 or 6 comes up, then drawer B is selected and a coin is drawn.

   (a) Find the probability of selecting a dime.

$$P(\text{dime}) = P(\text{dime}|A)P(A) + P(\text{dime}|B)P(B)$$
$$= \frac{3}{8}\cdot\frac{1}{3} + \frac{7}{10}\,\frac{2}{3}$$
$$= \frac{1}{8} + \frac{14}{30} = \frac{1}{8} + \frac{7}{15}$$

   (b) Suppose a dime is obtained. What is the probability that it came from drawer B?

Use Bayes' rule.

$$P(B|\text{dime}) = \frac{P(\text{dime}|B)P(B)}{P(\text{dime})}$$

$$= \frac{\frac{7}{10}\cdot\frac{2}{3}}{\frac{1}{8}+\frac{7}{15}} \leftarrow \text{from part a.}$$

2. A football coach has 49 players available for duty on a special kick-receiving team.

   (a) If eleven players must be chosen to play on this special team, how many different teams are possible?

   order does not matter
   sample w.o. replacement
   So, we use combination.

   $\binom{49}{11}$ possible teams

   (b) If the 49 include 28 offensive and 21 defensive players, what is the probability that a randomly selected team has 6 offensive and 5 defensive players?

   Hypergeometric
   2 groups -- taking a sample of 11 players w.o. replacement

   $$\frac{\binom{28}{6}\binom{21}{5}}{\binom{49}{11}}$$

3. If random variable X has the moment generating function

$$M(t) = e^{2t+3t^2},$$

find the variance of X using that moment generating function.

$$M'(t) = (2+6t)e^{2t+3t^2}$$

$$M''(t) = 6e^{2t+3t^2} + (2+6t)^2 e^{2t+3t^2}$$

$$E[X] = M'(0) = 2$$

$$E[X^2] = M''(0) = 6 + 2^2 = 10$$

$$Var[X] = E[X^2] - (E[X])^2$$
$$= 10 - 4$$
$$= 6$$

4. Find $E[z^X]$ where z is a real number and X is a geometric-distributed random variable. (Hint: Recall how we calculated the moment generating function for a geometric-distributed random variable.)

note
$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

$$f(x) = q^{x-1} p$$
$$x = 1, 2, \ldots$$

$$E[z^X] = \sum_{x=1}^{\infty} z^x q^{x-1} p = zp \sum_{x=1}^{\infty} z^{x-1} q^{x-1}$$

$$= zp \sum_{x=1}^{\infty} (zq)^{x-1}$$

$$= \frac{zp}{1-zq}$$

(as long as $|zq| < 1$)

5. A machine makes bolts at the rate of 1000 per hour. The probability of any given bolt is defective is 0.023 and is independent of whether any other bolt is defective.

   (a) What is the name of the distribution for the number of defects in an hour? What is the probability of 3 defects in an hour?

   Binomial

   $$P(X=3) = \binom{1000}{3} (0.023)^3 (1-0.023)^{1000-3}$$

   (b) Is there a standard way to approximate the probability of 3 defects in an hour given the circumstances? Write down this approximate probability.

   The law of rare events. n is large p is small

   Binomial is approximated by Poisson w. $\lambda = np$

   $$P(X=3) \approx \frac{(1000(0.023))^3 e^{-1000(0.023)}}{3!}$$

   (c) What is the mean number of defects in one hour? Is it different if you use the exact versus approximate distribution?

   The mean for X is $np = 1000 \cdot 0.023 = 23$.

   The poisson approximation uses $\lambda = np$ -- λ is the mean for a Poisson. The means are the same.

6. Let $E[X] = 2$ $Var[X] = 4$. Find

$$E\left[\left(\frac{X-2}{2}\right)^2\right]$$

$$E\left[\left(\frac{X-2}{2}\right)^2\right] = \frac{1}{4} E\left[(X-2)^2\right]$$

$$= \frac{1}{4} Var[X]$$ since 2 is the mean

$$= \frac{1}{4} \cdot 4$$

$$= 1$$

Math/Stat 414.2 Exam 1 Name: ____________________

Instructions: Please read!

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2. Show all work for full credit! Small mistakes in arithmetic wont reduce credit if you show work; but conversely, even a correct answer could get no credit without supporting work.
3. I will award partial credit where appropriate.

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1. Drawer A contains three pennies and five dimes, while drawer B contains seven pennies and three dimes. A drawer is selected in the following way. A die is rolled. If a 3,4,5 or 6 comes up, then drawer A is selected, and a coin is drawn. If a 1 or 2 comes up, then drawer B is selected, and a coin is drawn.

   (a) Find the probability of selecting a penny.

   $$P(\text{penny}) = P(\text{penny}|A)P(A) + P(\text{penny}|B)P(B)$$
   $$= \frac{3}{8}\frac{2}{3} + \frac{7}{10}\frac{1}{3}$$
   $$= \frac{1}{4} + \frac{7}{30}$$

   (b) Suppose a dime is obtained. What is the probability that it came from drawer A?

   Use Bayes' rule

   $$P(A|\text{dime}) = \frac{P(\text{dime}|A)P(A)}{P(\text{dime})} = \frac{\frac{5}{8}\cdot\frac{2}{3}}{\frac{5}{12}+\frac{1}{10}}$$

2. A basketball coach has 20 players available for the next game.

   (a) If 5 players must be chosen to play, how many different teams are possible?

   order does not matter
   sample w.o. replacement
   We use combination.

   $\binom{20}{5}$ possible teams

   (b) If the 20 players include 12 frontcourt and 8 backcourt players, what is the probability that a randomly selected team has 3 frontcourt and 2 backcourt players?

   Hypergeometric
   2 groups
   take a sample of 5 players w.o. replacement

   $$\frac{\binom{12}{3}\binom{8}{2}}{\binom{20}{5}}$$

3. If random variable X has the moment generating function $M(t) = e^{2t^2}$, then find the variance of X.

$M(t) = e^{2t^2}$

$M'(t) = 4te^{2t^2}$

$M''(t) = 16t^2 e^{2t^2} + 4e^{2t^2}$

$E[X] = M'(0) = 0$

$E[X^2] = M''(0) = 4$

$Var[X] = E[X^2] - (E[X])^2$

$= 4 - 0$

$= 4$

4. Find $E[z^X]$ where z is a real number and X is a Poisson-distributed random variable. (Hint: Recall how we calculated the moment generating function for a Poisson random variable.)

$$E[z^X] = \sum_{x=0}^{\infty} z^x \frac{\lambda^x e^{-\lambda}}{x!}$$

$$= \sum_{x=0}^{\infty} \frac{(\lambda z)^x e^{-\lambda}}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda z)^x}{x!}$$

$$= e^{-\lambda} e^{\lambda z}$$

$$= e^{\lambda(z-1)}$$

5. A machine in our factory makes bolts. To test whether a bolt is defective, it must be destroyed. The probability that any given bolt is defective is 0.023 and is independent of whether any other bolt is defective.

(a) What is the name of the distribution for the number of bolts destroyed to find the first defect?

geometric -- number of trials to first success

(b) What is the probability that fewer than 3 tests are needed to find the first defect?

$$P(X \leq 3) = \sum_{x=1}^{3} (1-0.023)^{x-1} (0.023)$$

$$= 0.023 + (1-0.023)(0.023) + (1-0.023)^2 (0.023)$$

(c) If each bolt costs the company \$0.01 and the cost to test each bolt is \$0.02, what is the expected cost of finding that defect?

Cost per bolt is \$0.03. Total expected cost is $E[0.03X] = 0.03\,E[X]$

$$= \frac{0.03}{0.023}$$

6. Let $E[X] = 3$ and $Var[X] = 4$. Find

$$E\left[\left(\frac{X-3}{2}\right)^2\right].$$

$$E\left[\left(\frac{X-3}{2}\right)^2\right]$$

$$= \frac{1}{4} E[(X-3)^2]$$

$$= \frac{1}{4} Var[X]$$

$$= \frac{1}{4} \cdot 4$$

$$= 1$$