Math/Stat 414 Exam 2 Name: ____________

Instructions: Please read!

1. Do all work on this exam packet. It is okay to leave your answer unsimplified, as in $0.56\frac{15!}{6!9!}$ or $14e - 20$. This means that no calculator is needed.
2. Show all work for full credit! Small mistakes in arithmetic will not reduce credit if you show work; conversely, even a correct answer could get no credit without supporting work.
3. I will award partial credit where appropriate.

---

1. Let X have a cdf of

$$F(x) = 1 - e^{-2x}$$

for $x > 0$ (and 0 when $x \leq 0$).

(a) Find the $P(X > 0.5|X \leq 2)$.
solution: The density for X is $f(x) = 2e^{-2x}$ for non-negative x.

$$P(X > 0.5|X \leq 2) = \frac{P(0.5 < X \leq 2)}{P(X \leq 2)} = \frac{\int_{0.5}^{2} 2e^{-2x}dx}{\int_{0}^{2} 2e^{-2x}dx}$$

Solving the integrals, we have

$$P(X > 0.5|X \leq 2) = \frac{e^{-1} - e^{-4}}{1 - e^{-4}}$$

(b) Find Ee^X.

solution:

$$Ee^X = \int_0^\infty e^x 2e^{-2x} dx = \int_0^\infty 2e^{-x} dx = 2\int_0^\infty e^{-x} dx = 2$$

(c) Find the density for

$$Y = \frac{1}{\sqrt{X}}$$

solution: We use the CDF method. For $y > 0$

$$F_Y(y) = P(Y \leq y) = P(1/\sqrt{X} \leq y) = P(1/y \leq \sqrt{X}) = P(1/y^2 \leq X) = 1 - P(X \leq 1/y^2)$$

So,

$$F_Y(y) = 1 - F_X(1/y^2) = e^{-2/y^2}$$

To find the density, we take the derivative with respect to y of the CDF by applying the chain rule.

$$f_Y(y) = F_Y'(y) = 4y^{-3}e^{-2/y^2}$$

for $y > 0$.

2. Suppose X and Y have a joint density

$$f(x, y) = cI_{\{x^2+y^2<1, 0<x<1, 0<y<1\}}$$

for some constant $c > 0$. In other words, X and Y are uniformly distributed in the unit quarter-disk of points in the positive quadrant.

(a) Find c. (You may use integration or facts from geometry. Just be clear about your derivation.)

solution: The support of this uniform distribution is one fourth of a unit circle. The area of a circle with radius one is π. Therefore, the area of the support is $\pi/4$. So, $c = 4/\pi$.

(b) Find the marginal distribution of X.

solution: We need to integrate the joint density over y. For x between zero and one,

$$f_X(x) = \int_0^{\sqrt{1-x^2}} \frac{4}{\pi} dy = \frac{4}{\pi}\sqrt{1-x^2}$$

(c) Find the $P(X+Y>1)$.

solution: The easiest way to do this problem is to take the entire area of the quarter disk ($\pi/4$) and subtract the right triangle bounded by the line $y=1-x$ (this triangle has area $1/2$). The area of this region is thus $\pi/4-1/2$. We then divide by the constant to ensure it is a probability. So, $P(X+Y>1)=\frac{\pi/4-1/2}{\pi/4}$

3. Three fair coins are tossed. Let X be the number of heads in the three tosses. Let Y be the number of tails that precede the first head.

   (a) Find the joint distribution of X and Y.

   solution: The first row represents the possible values for X and the left column represents the possible values for Y. The entries in the table represent the probabilities of particular outcomes along with corresponding points in the sample space.

	0	1	2	3
0	0	$1/8(HTT)$	$2/8(HTH, HHT)$	$1/8(HHH)$
1	0	$1/8(THT)$	$1/8(THH)$	0
2	0	$1/8(TTH)$	0	0
3	$1/8(TTT)$	0	0	0

   (b) Find the conditional distribution of X given Y.

   solution: With the table above, the conditional distribution of X given Y will be the rows dividing by the total of that row, so that it is a probability distribution. In other words for $Y = 0$, the conditional distribution $p_{X|Y}(x|0)$ is

	0	1	2	3
	0	$1/4(HTT)$	$2/4(HTH, HHT)$	$1/4(HHH)$

For $Y = 1$,

	0	1	2	3
	0	$1/2(THT)$	$1/2(THH)$	0

For $Y = 2$,

	0	1	2	3
	0	$1(TTH)$	0	0

For $Y = 3$,

	0	1	2	3
	$1(TTT)$	0	0	0

4. Let X and Y be independent with each having a uniform distribution on $(0,1)$. Find the density for $U = max(X,Y)$.

   solution: We use the cdf method again. For $0 < u < 1$,

$$F_U(u) = P(U \leq u) = P(max(X,Y) \leq u) = P(X \leq u, Y \leq u) = P(X \leq u)P(Y \leq u) = u^2$$

   Note that we may separate the intersection into a product since X and Y are independent. Also, $P(X \leq u) = u$ since X is standard uniform. We now take the derivative with respect to u. So,

$$f_U(u) = 2uI_{\{0<u<1\}}$$