STAT/MATH 414, section 001
Final Exam, Tuesday, May 3

Name: ____

- Do all work on this exam packet; there is extra space on the back page if needed. It is okay to leave your answer unsimplified, as in $0.5^6 \frac{15!}{6!9!}$ or $14e^{-20}$.
- Show all work for full credit! Small mistakes in arithmetic won't reduce credit if you show work; but conversely, even a correct answer could get no credit without supporting work.
- I will award partial credit where appropriate.
- When a question asks for an answer to three decimal places, the computations when done correctly are straightforward and will not require a calculator.

$$P(Z \le z) = \Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-w^2/2}\, dw$$

$$\Phi(-z) = 1 - \Phi(z)$$

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990

1. A pocket contains three coins, one of which has a head on both sides, while the other two coins are normal. A coin is chosen at random from the pocket and tossed three times.

   (a) Find the probability of obtaining three heads.

   solution: Let A be the event of pulling the double sided coin. Note that $P(A) = 1/3$. By the law of total probability:

   $$P(HHH) = P(HHH|A)P(A) + P(HHH|A^c)P(A^c) = 1(1/3) + 1/8(2/3) = 5/12$$

   (b) If the head turns up all three times, what is the probability that this is the two-headed coin?

   solution: We apply Bayes theorem:

   $$P(A|HHH) = \frac{P(HHH|A)P(A)}{P(HHH|A)P(A) + P(HHH|A^c)P(A^c)} = \frac{1/3}{5/12} = \frac{4}{5}$$

2. Let X and Y be discrete random variables with joint pmf

$$f(x,y) = \frac{4}{5(x+1)(y+2)}$$

if $x = 0, 1$ and $y = 0, 1$, and zero otherwise.

(a) Find the $Cov(X,Y)$.
solution: We first need to find EX and EY.

$$EX = 0P(X=0)+1P(X=1) = P(X=1) = P(X=1,Y=0)+P(X=1,Y=1) = \frac{4}{20}+\frac{4}{30} = \frac{1}{3}$$

and

$$EY = 0P(Y=0)+1P(Y=1) = P(Y=1) = P(X=0,Y=1)+P(X=1,Y=1) = \frac{4}{15}+\frac{4}{30} = \frac{12}{30} = \frac{2}{5}$$

Now, the cross moment

$$EXY = \sum_{x=0,1\ y=0,1} xyP(X=x,Y=y) = P(X=1,Y=1) = \frac{4}{30}$$

So,

$$Cov(X,Y) = EXY - EXEY = \frac{2}{15} - \frac{2}{15} = 0$$

(b) Find the variance of $X+Y$.
solution:

$$Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y)$$

Since X and Y are marginally Bernoulli, the second moment is equal to the first. So,

$$Var(X) = \frac{1}{3} - \frac{1}{9} = \frac{2}{9}$$

and

$$Var(Y) = \frac{2}{5} - \frac{4}{25} = \frac{6}{25}$$

and

$$Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y) = \frac{2}{9} + \frac{6}{25} + 0$$

3. Suppose that X and Y are independent random variables with densities

$$f_X(x) = ce^x,\ 0 < x < 3$$

and

$$f_Y(y) = ce^y,\ 0 < y < 3$$

(a) Find c.
solution:

$$1 = c\int_0^3 e^x dx$$

So,

$$c = \frac{1}{e^3 - 1}$$

(b) Find the moment generating function of X. Use this to find the moment generating function of $X + Y$.
solution:

$$M_X(t) = c\int_0^3 e^{xt}e^x dx = \frac{c}{t+1}\int_0^3 e^{x(t+1)}(t+1)dx = \frac{e^{3(t+1)} - 1}{(t+1)(e^3 - 1)}$$

Since Y has the same distribution and is independent of X, the MGF of their sum is the product of the individual MGF's. So,

$$M_{X+Y}(t) = \left(\frac{e^{3(t+1)} - 1}{(t+1)(e^3 - 1)}\right)^2$$

(c) Find the density of $X+Y$ using the convolution method.

$$f_{X+Y}(z) = \int f_X(z-y) f_Y(y) dy$$

for $0 < z < 6$. Like the uniform example, we break up this interval into above and below 3. For $0 < z < 3$,

$$f_{X+Y}(z) = \int_0^z f_X(z-y) f_Y(y) dy = c^2 \int_0^z e^{z-y} e^y dy = c^2 y e^z \Big|_0^z = c^2 z e^z$$

For $3 \le z < 6$,

$$f_{X+Y}(z) = \int_{z-3}^3 f_X(z-y) f_Y(y) dy = c^2 \int_{z-3}^3 e^{z-y} e^y dy = c^2(3-(z-3))e^z = c^2(6-z)e^z$$

4. A bus has capacity for twenty passengers. After the opening of ticket sales, the time in minutes between sales of tickets has mean 10 and variance 20.

   (a) Give an upper bound on the probability that the total time to sell out of tickets is within thirty minutes of two hundred minutes.

   solution:

   Let X_i be the amount of time between the $i-1$ st ticket and the ith ticket being bought. Now, $\sum_{i=1}^{20} X_i$ is the time that the 20th ticket is bought. We may use Chebyshev's inequality now.

$$P(|\sum_{i=1}^{20} X_i - 200| > 30) \leq \frac{400}{900}$$

   So, a lower bound can be created by taking the complement

$$P(|\sum_{i=1}^{20} X_i - 200| < 30) \geq 1 - \frac{4}{9}$$

   (b) Now, further assume that the distribution of time in minutes between sales of tickets is gamma with $\alpha = 5$ and $\lambda = 1/2$. Assuming independence between times, give an integral representation for the probability that the time to run out of tickets will be less than four hours.

   solution: Since the sum of gamma random variables is gamma, $\sum_{i=1}^{20} X_i$ is gamma with $\alpha = 100$ and $\lambda = 1/2$. So, the probability can be written as

$$P(\sum_{i=1}^{20} X_i \leq 240) = \int_0^{240} \frac{(0.5)^{100} x^{99} e^{-0.5x}}{\Gamma(100)} dx$$

   (c) Approximate the probability (to three decimal points) that the time to run out of tickets will be less than four hours.

$$P\left(\sum_{i=1}^{20} X_i \leq 240\right) = P\left(\frac{\sum_{i=1}^{20} X_i - 200}{\sqrt{2020}} \leq \frac{240-200}{\sqrt{2020}}\right) = \Phi\left(\frac{2}{\sqrt{20}}\right)$$

   This is fine for a final answer.

5. Suppose that X and Y are jointly continuous with density

$$f(x, y) = \frac{1}{8}(x + y + 2), \ -1 < x < 1, \quad -1 < y < 1$$

(a) Find the marginal density of Y.

solution: Integrate over x.

$$f_Y(y) \int_{-1}^{1} \frac{1}{8}(x + y + 2)\, dx = \frac{1}{8}\left(\frac{1}{2}x^2 + xy + 2x\right)\Big|_{x=-1}^{x=1} = \frac{1}{4}(y + 2)$$

for $-1 < y < 1$.

(b) Find the conditional density of X given Y. Use this to write down the integral representing $P(X < 0|Y = 0)$.

solution: The conditional density is

$$f_{X|Y}(x|y) = \frac{f(x, y)}{f_Y(y)} = \frac{\frac{1}{8}(x + y + 2)}{\frac{1}{4}(y + 2)} = \frac{x + y + 2}{2(y + 2)}$$

for $-1 < x < 1, -1 < y < 1$.

So,

$$P(X < 0|Y = 0) = \int_{-1}^{0} \frac{x + 2}{4} dx$$

(c) Find the density of $W = Y^2$.
solution: We use the cdf method
For $0 < w < 1$,

$$P(W \leq w) = P(Y^2 \leq w) = P(-\sqrt{w} \leq Y \leq \sqrt{w}) = \int_{-\sqrt{w}}^{\sqrt{w}} \frac{1}{4}(y+2)dy = \sqrt{w}$$

Now, take the derivative with respect to w. We have

$$\frac{1}{2}\frac{1}{\sqrt{w}}$$

6. NOTE:We did not cover the material in this question! Suppose that X has density

$$f(x) = \frac{3}{2}x - \frac{3}{4}x^2, \ 0 < x < 2$$

(a) Use the inversion method to specify a technique to generate X.

(b) Use the accept-reject method to specify a technique to generate X. Use a uniform from 0 to 2 as your proposal distribution. (We have been using the letter Y to denote the proposal and using $g(y)$ to denote its density.)