Math/Stat 414.1 Exam 2 Name: ____________

Instructions: Please read!

1. Do all work on this exam packet. It is okay to leave your answer unsimplified, as in $0.56\frac{15!}{6!9!}$ or $14e - 20$. This means that no calculator is needed.
2. Show all work for full credit! Small mistakes in arithmetic wont reduce credit if you show work; conversely, even a correct answer could get no credit without supporting work.
3. I will award partial credit where appropriate.

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1. Random variable X which has density

$$f(x) = \frac{1}{2}(x^3 + 1), \quad -1 < x < 1$$

$$E[X] = \int_{-1}^{1} \frac{1}{2}(x^4 + x)\,dx = \left[\frac{1}{10}x^5 + \frac{1}{4}x^2\right]_{x=-1}^{x=1} = \frac{1}{10} + \frac{1}{10} = \frac{2}{10} = \frac{1}{5}$$

(a) Find the variance of X

$$E[X^2] = \int_{-1}^{1} \frac{1}{2}x^5 + \frac{1}{2}x^2\,dx = \left[\frac{1}{12}x^6 + \frac{1}{6}x^3\right]_{x=-1}^{x=1} = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

$$Var[X] = E[X^2] - (E[X])^2 = \frac{1}{3} - \left(\frac{1}{5}\right)^2$$

(b) Find the density of $Y = X^2$. Be sure to note the support of the resulting density.

$$F_Y(y) = P(Y \le y) = P(X^2 \le y) = P(-\sqrt{y} < X < \sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y}),$$
$$0 < y < 1$$

$$f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{d}{dy}\left(F_X(\sqrt{y}) - F_X(-\sqrt{y})\right) = f_X(\sqrt{y})\left(\frac{1}{2}y^{-1/2}\right) + f_X(-\sqrt{y})\left(\frac{1}{2}y^{-1/2}\right)$$

$$= \frac{1}{2}\left(y^{3/2} + 1\right)\frac{1}{2}y^{-1/2} + \frac{1}{2}\left(-y^{3/2} + 1\right)\frac{1}{2}y^{-1/2}$$

$$= \frac{1}{2}y^{-1/2}$$

2. Let T be uniform in the interval $(0, 2)$ and S be independent of T and also uniform in the interval $(0, 2)$.

   (a) Find $P(S(T+1) < 1)$

$$P(S(T+1)<1) = P(S < \frac{1}{T+1})$$

$$= \int_0^2 \int_0^{\frac{1}{t+1}} \frac{1}{2} \cdot \frac{1}{2} \, ds \, dt$$

$$= \frac{1}{4} \int_0^2 \frac{1}{t+1} \, dt$$

$$= \frac{1}{4} \left[\log(t+1) \right]_{t=0}^{t=2}$$

$$= \frac{1}{4} \left[\log(3) - \log(1) \right] = \frac{\log 3}{4}$$

   (b) Find the moment generating function for T. SHOW YOUR WORK. This means you need to start with the definition of the moment generating function and showing the steps to arrive at the final answer.

$$M_T(u) = \frac{1}{2} \int_0^2 e^{ut} \, dt$$

$$= \frac{1}{2u} \int_0^2 e^{ut} \, u\,dt$$

$$= \frac{1}{2u} \left[e^{ut} \right]_{t=0}^{t=2}$$

$$= \frac{e^{2u} - 1}{2u}$$

(c) Find the moment generating function for $T + S$. Does this correspond to the moment generating function for a distribution we know?

$$M_{T+S}(u) = M_T(u)\, M_S(u) \quad \leftarrow \text{independence of } S, T$$

$$= \frac{e^{2u}-1}{u} \; \frac{e^{2u}-1}{u} \quad \leftarrow \text{Known from part b.}$$

$$= \left(\frac{e^{2u}-1}{2u} \right)^2$$

No, this is not one of the MGF's we've studied.

3. Let X have density

$$f_X(x) = \frac{3}{8}x^2, \quad 0 < x < 2 \quad \rightarrow \quad f_X(x) = \frac{3}{8}x^2 \, I_{\{0<x<2\}}$$

and Y be independent of X with density

$$f_Y(y) = \frac{3}{8}y^2, \quad 0 < y < 2$$

Find the density of the sum $X + Y$ using the convolution method.

$$f_{X+Y}(a) = \int_{-\infty}^{\infty} f_X(a-y)\, f_Y(y)\, dy$$

$$= \int_{-\infty}^{\infty} \frac{3}{8}(a-y)^2 \, \frac{3}{8} y^2 \, \underbrace{I_{\{0<a-y<2\}}}_{I_{\{a-2<y<a\}}} I_{\{0<y<2\}} \, dy$$

For $0<a<2$

$$f_{X+Y}(a) = \int_0^a \frac{9}{64}(a-y)^2 y^2 \, dy = \frac{9}{64}\int_0^a y^4 - 2ay^3 + a^2y^2 \, dy$$

$$= \frac{9}{64}\left[\frac{1}{5}y^5 - \frac{1}{2}ay^4 + \frac{1}{3}a^2y^3\right]_0^a$$

$$= \frac{9}{64}\left[\frac{1}{5}a^5 - \frac{1}{2}a^5 + \frac{1}{3}a^5\right]$$

$$= \frac{9}{64}\left[\frac{1}{30}a^5\right]$$

For $2 \le a < 4$

$$f_{X+Y}(a) = \int_{a-2}^{2} \frac{9}{64}(y^4 - 2ay^3 + a^2y^2)\, dy = \frac{9}{64}\left[\frac{1}{5}y^5 - \frac{1}{2}ay^4 + \frac{1}{3}a^2y^3\right]_{a-2}^{2}$$

$$= \frac{9}{64}\left(\frac{1}{5}2^5 - \frac{1}{2}a(2^4) + \frac{1}{3}a^2(2^3) - \frac{1}{5}(a-2)^5 - \frac{1}{2}a(a-2)^4 + \frac{1}{3}a^2(a-2)^3\right)$$

4. Let X and Y have joint density

$$f(x,y) = \frac{12}{5}y(2-y-x), \quad 0<x<1, \quad 0<y<1$$

need

$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$

(a) Find $P\left(X > \frac{1}{2} \middle| Y = \frac{1}{2}\right)$.

Find

$$f_Y(y) = \int_0^1 \frac{12}{5}\, y(2-y-x)\, dx$$

$$= \left[\frac{12}{5}\left(2xy - xy^2 - \tfrac{1}{2}x^2y\right)\right]_{x=0}^{1}$$

$$= \frac{12}{5}\left(2y - y^2 - \tfrac{1}{2}y\right)$$

$$= \frac{12}{5}\left(\tfrac{3}{2}y - y^2\right)$$

$$P(X>\tfrac{1}{2}|Y=\tfrac{1}{2}) = \int_{1/2}^{1} f_{X|Y}(x|y=\tfrac{1}{2})\, dx$$

$$= \int_{1/2}^{1} \frac{f(x,\frac{1}{2})}{f_Y(\frac{1}{2})}\, dx$$

$$= \int_{1/2}^{1} \frac{\cancel{\frac{12}{5}}\cancel{\frac{1}{2}}(2-\frac{1}{2}-x)}{\cancel{\frac{12}{5}}\left(\cancel{\frac{3}{4}-\frac{1}{4}}\right)}\, dx$$

$$= \int_{1/2}^{1} \left(\tfrac{3}{2} - x\right) dx$$

$$= \tfrac{3}{2}x - \tfrac{1}{2}x^2 \Big|_{1/2}^{1}$$

$$= \tfrac{3}{2} - \tfrac{1}{2} - \left(\tfrac{3}{2}\cdot\tfrac{1}{2} - \tfrac{1}{2}\cdot\tfrac{1}{4}\right)$$

$$= 1 - \left(\tfrac{6}{8} - \tfrac{1}{8}\right)$$

$$= \tfrac{3}{8}$$

(b) Find $E[XY]$.

$$E[XY] = \int_0^1\int_0^1 \frac{12}{5}\, xy^2(2-y-x)\, dx\, dy$$

$$= \int_0^1\int_0^1 \frac{12}{5}\left(2xy^2 - xy^3 - x^2y^2\right) dx\, dy$$

$$= \int_0^1 \left[\frac{12}{5}\left(x^2y^2 - \tfrac{1}{2}x^2y^3 - \tfrac{1}{3}x^3y^2\right)\right]_{x=0}^{x=1} dy$$

$$= \int_0^1 \frac{12}{5}\left(y^2 - \tfrac{1}{2}y^3 - \tfrac{1}{3}y^2\right) dy$$

$$= \frac{12}{5}\left[\tfrac{2}{9}y^3 - \tfrac{1}{8}y^4\right]_{y=0}^{y=1}$$

$$= \frac{12}{5}\left[\frac{2}{9} - \frac{1}{8}\right]$$

$$= \frac{12}{5}\left(\frac{16}{72} - \frac{9}{72}\right) = \frac{12}{5}\,\frac{7}{72} = \frac{7}{30}$$

Math/Stat 414.2 Exam 2 Name: ______________________

Instructions: Please read!

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2. Show all work for full credit! Small mistakes in arithmetic wont reduce credit if you show work; but conversely, even a correct answer could get no credit without supporting work.
3. I will award partial credit where appropriate.

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1. Random variable X which has density

$$f(x) = \frac{1}{8}x + \frac{2}{8}, \quad -2 < x < 2$$

(a) Find the variance of X.

$$E[X] = \int_{-2}^{2} x\left(\frac{1}{8}x + \frac{2}{8}\right)dx = \frac{1}{24}x^3 + \frac{1}{8}x^2 \Big|_{-2}^{2} = \frac{2}{3}$$

$$E[X^2] = \int_{-2}^{2} x^2\left(\frac{1}{8}x + \frac{2}{8}\right)dx = \frac{1}{32}x^4 + \frac{2}{24}x^3 \Big|_{-2}^{2} = \frac{4}{3}$$

$$Var[X] = E[X^2] - E[X]^2 \quad \frac{4}{3} - \frac{4}{9} = \frac{8}{9}$$

(b) Find the density of $Y = X^2$. Be sure to note the support of the resulting density.

$$0 < Y < 2$$

$$F_Y(y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y})$$
$$= F_X(\sqrt{y}) - F(-\sqrt{y})$$

$$f_Y(y) = \frac{d}{dy}\left(F_X(\sqrt{y}) - F_X(-\sqrt{y})\right) = \frac{1}{2}y^{-1/2} f_X(\sqrt{y}) + \frac{1}{2}y^{-1/2} f_X(-\sqrt{y})$$

$$= \frac{1}{2}y^{-1/2}\left(\frac{1}{8}y^{1/2} + \frac{2}{8}\right) + \frac{1}{2}y^{-1/2}$$

$$\left(\frac{1}{8}(-\sqrt{y}) + \frac{2}{8}\right)$$

$$= \frac{1}{8}y^{-1/2} + \frac{1}{8}y^{-1/2}$$
$$= \frac{1}{4}y^{-1/2}$$

2. Let T be exponential with mean two and S be independent of T and exponential with mean two.

   (a) Find $P(T - S > 2)$.

$$P(T-S>2) = P(T>S+2)$$
$$= \int_0^\infty \int_{S+2}^\infty \frac{1}{4} e^{-t/2} e^{-s/2}\, dt\, ds$$
$$= \int_0^\infty -\frac{1}{2} e^{-s/2} \int_{s+2}^\infty -e^{-t/2}\frac{1}{2} dt\, ds$$
$$= \int_0^\infty -\frac{1}{2} e^{-s/2} \left[e^{-t/2}\right]_{t=s+2}^{t=\infty} ds$$
$$= \int_0^\infty \frac{1}{2} e^{-s/2} e^{-(s+2)/2}\, ds$$
$$= \frac{1}{2} e^{-1} \int_0^\infty e^{-s}\, ds = \frac{1}{2} e^{-1}$$

   (b) Find the moment generating function for T. SHOW YOUR WORK. This means you need start with the definition of the moment generating function and showing the steps to arrive at the final answer.

$$M_T(u) = \int_0^\infty e^{tu} \frac{1}{2} e^{-t/2}\, dt$$
$$= \frac{1}{2} \int_0^\infty e^{t(u-\frac{1}{2})}\, dt$$
$$= \frac{1}{2} \frac{1}{u-\frac{1}{2}} \int_0^\infty e^{t(u-\frac{1}{2})} (u-\tfrac{1}{2})\, dt$$
$$= \frac{1}{2} \frac{1}{u-\frac{1}{2}} e^{t(u-\frac{1}{2})} \Big|_0^\infty$$
$$= \frac{1}{2u-1}(0-1) \quad \text{(as long as } u<\tfrac{1}{2}$$

$$= \frac{1}{1-2u}$$

(c) Find the moment generating function for $T + S$. Does this correspond to the moment generating function for a distribution we know?

$$M_{T+S}(u) = M_T(u)\, M_S(u)$$ since S,T are independent.

$$= \left(\frac{1}{1-2u}\right)\left(\frac{1}{1-2u}\right)$$ The MGF's are the same since S,T have same distribution.

$$= \left(\frac{1}{1-2u}\right)^2$$

This is the MGF corresponding to a gamma distribution

3. Let X have density

$$f_X(x) = \frac{1}{2}x, \quad 0 < x < 2 \quad \rightarrow f_X(x) = \frac{1}{2}x\, 1_{\{0<x<2\}}$$

and Y be independent of X with density

$$f_Y(y) = \frac{1}{2}y, \quad 0 < y < 2 \quad \rightarrow f_Y(y) = \frac{1}{2}y\, 1_{\{0<y<2\}}$$

Find the density of the sum $X + Y$ using the convolution method.

$$f_{X+Y}(a) = \int_{-\infty}^{\infty} f_X(a-y) f_Y(y)\, dy$$

$$= \int_{-\infty}^{\infty} \frac{1}{4}(a-y)y \underbrace{1_{\{a-2<y<a\}} 1_{\{0<y<2\}}}\, dy$$

$$= 1_{\{0<y<a\}} \text{ for } a<2$$

$$= 1_{\{a-2<y<2\}} \text{ for } a \geq 2$$

For $a<2$

$$f_{X+Y}(a) = \int_0^a \frac{1}{4}(a-y)y\, dy = \frac{1}{8}ay^2 - \frac{1}{12}y^3 \Big|_0^a$$

$$= \frac{1}{8}a^3 - \frac{1}{12}a^3 = \frac{1}{24}a^3$$

For $a \geq 2$

$$f_{X+Y}(a) = \int_{a-2}^{2} \frac{1}{4}(a-y)y\, dy = \frac{1}{8}ay^2 - \frac{1}{12}y^3 \Big|_{a-2}^{2}$$

$$= \frac{1}{2}a - \frac{2}{3} - \frac{1}{8}a(a-2)^2 + \frac{1}{12}(a-2)^3$$

$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$ 4. Let X and Y have joint density

so Find $\downarrow$ $f_Y(y)$

$$f(x,y) = \frac{12}{5}x(2-x-y), \quad 0<x<1, \quad 0<y<1$$

(a) Find $P\left(X > \frac{1}{2}\middle| Y = \frac{1}{2}\right)$.

$$= \int_0^1 \frac{12}{5}(2x - x^2 - xy)\,dx$$

$$= \frac{12}{5}\left[x^2 - \frac{1}{3}x^3 - \frac{1}{2}x^2 y\right]_{x=0}^{x=1}$$

$$= \frac{12}{5}\left[\frac{2}{3} - \frac{1}{2}y\right]$$

$$P(X > \tfrac{1}{2} | Y = \tfrac{1}{2}) = \int_{1/2}^1 f_{X|Y}(x|\tfrac{1}{2})\,dx$$

$$= \int_{1/2}^1 \frac{x(2-x-\frac{1}{2})}{\frac{2}{3}-\frac{1}{4}}\,dx$$

$$= \frac{12}{5}\int_{1/2}^1 \frac{3}{2}x - x^2\,dx$$

$$= \frac{12}{5}\left[\frac{3}{4}x^2 - \frac{1}{3}x^3\right]_{x=1/2}^{x=1}$$

$$= \frac{12}{5}\left[\frac{3}{4} - \frac{1}{3} - \frac{3}{16} + \frac{1}{24}\right]$$

$$= \frac{13}{20}$$

(b) Find $E[XY]$.

$$E[XY] = \int_0^1\int_0^1 \frac{12}{5}x^2 y(2-x-y)\,dx\,dy$$

$$= \int_0^1\int_0^1 \frac{12}{5}(2x^2y - x^3y - x^2y^2)\,dx\,dy$$

$$= \frac{12}{5}\int_0^1\left[\frac{2}{3}x^3y - \frac{1}{4}x^4y - \frac{1}{3}x^3y^2\right]_{x=0}^{x=1}dy$$

$$= \frac{12}{5}\int_0^1 \frac{5}{12}y - \frac{1}{3}y^2\,dy$$

$$= \frac{12}{5}\left[\frac{5}{24}y^2 - \frac{1}{9}y^3\right]_{y=0}^{y=1}$$

$$= \frac{12}{5}\left[\frac{5}{24} - \frac{1}{9}\right] = \frac{7}{30}$$

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