Math/Stat 416 Quiz 1 Name: ____________________

**Instructions: Please read!**

1. Do all work on this exam packet. It is okay to leave your answer unsimplified, as in $0.56\frac{15!}{6!9!}$ or $14e - 20$. This means that no calculator is needed.
2. Show all work for full credit! Small mistakes in arithmetic will not reduce credit if you show work; conversely, even a correct answer could get no credit without supporting work.
3. I will award partial credit where appropriate.

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1. (25 pts) *Show the derivation* of the moment generating function for a Poisson random variable. The probability mass function (which you should know by memory) is

$$f(x) = \frac{\lambda^x e^{-\lambda}}{x!},\ x = 0, 1, 2, ...$$

(So that there is no confusion, the point of this problem is to SHOW the derivation–not to state the answer–which is below!) Use the derived moment generating function

$$M(t) = e^{\lambda(e^t - 1)}$$

to show that the mean of a Poisson is $\lambda$.

*solution:*

$$M(t) = Ee^{tX} = \sum_{x=0}^{\infty} e^{tx}\frac{\lambda^x e^{-\lambda}}{x!} = e^{-\lambda}\sum_{x=0}^{\infty}\frac{(\lambda e^t)^x}{x!}$$

Note that this last is a power series for the exponential function. So,

$$M(t) = e^{\lambda(e^t - 1)}$$

To use this to find the mean, we evaluate the first derivative of the MGF at zero.

$$M'(t) = \lambda e^t e^{\lambda(e^t - 1)}$$

So,

$$M'(0) = \lambda$$

2. (25 pts) If a fair coin is successively flipped, find the probability that a head first appears on the fifth trial. What if the coin was biased such that the probability of a head appearing were 0.6; what would be the probability in this case also?

   *solution:* Assuming the flips are independent

$$P(TTTTH) = 0.5^4 0.5$$

   If the probability of a head is now 0.6, then

$$P(TTTTH) = 0.4^4 0.6$$

3. (25 pts) A coin that comes up heads with probability $p$ is continually flipped until the pattern $H, H, T$ appears. (That is, you stop flipping when the most recent flip lands tails and the two immediately preceding it lands heads.) Let $X$ denote the number of flips made. Find $E[X]$. You may leave your answer in terms of an equation in $E[X]$ and $p$. *solution:* Note that this problem is a modified version of one of your homework problems. Proceed using one-step arguments.

$$E[X] = E[X|H]p + E[X|T](1-p)$$

Note that if you get a $T$ on the first roll it is like starting over. So,

$$E[X] = E[X|H]p + (E[X]+1)(1-p)$$

Now,

$$E[X|H] = E[X|HH]p + E[X|HT](1-p)$$

We rewrite as

$$E[X|H] = E[X|HH]p + (E[X]+2)(1-p)$$

Now,

$$E[X|HH] = E[X|HHH]p + E[X|HHT](1-p)$$

This we rewrite as

$$E[X|HH] = (E[X|HH]+1)p + 3(1-p)$$

So,

$$E[X|HH] = \frac{p}{1-p} + 3$$

Substituting

$$E[X|H] = (\frac{p}{1-p} + 3)p + (E[X]+2)(1-p)$$

So,

$$E[X] = (\frac{p}{1-p} + 3)p^2 + (E[X]+2)(1-p)p + (E[X]+1)(1-p)$$

4. (25 pts) Suppose that $P(X = x|Y = y)$ is Poisson with rate $y$. In other words,

$$P(X = x|Y = y) = \frac{y^x e^{-y}}{x!}, \; y > 0 \; x = 0, 1, 2, ...,$$

Now, suppose that $Y$ has an exponential distribution with mean equal to two. Find the variance of $X$.

*solution:*

$$Var[X] = Var[E[X|Y]] + E[Var[X|Y]] = Var[Y] + E[Y]$$

This is due to the fact that the conditional distribution of $X$ given $Y$ is Poisson with parameter $Y$. Thus the conditional mean and variance are both $Y$. So,

$$Var[X] = 4 + 2 = 6$$

This is true since $Y$ is exponential with mean 2, thus its variance is 4.