Math/Stat 416 Quiz 2 Name: ______________________

**Instructions: Please read!**

1. Do all work on this exam packet. It is okay to leave your answer unsimplified, as in $0.56\frac{15!}{6!9!}$ or $14e - 20$. This means that no calculator is needed.
2. Show all work for full credit! Small mistakes in arithmetic will not reduce credit if you show work; conversely, even a correct answer could get no credit without supporting work.
3. I will award partial credit where appropriate.

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1. (25 points) Suppose that $X$ is an exponential random variable with mean 3. Further suppose that the conditional density of $Y$ given $X = x$ is exponential with mean $x$. In other words,

$$f_{Y|X}(y|x) = \frac{1}{x}e^{-y/x}$$

for $x, y > 0$. Find the variance of $Y$.

*solution:*

$$E(X^2) = (E[X])^2 + Var(X) = 3^2 + 9 = 18$$

Now by the law of total variance,

$$Var(Y) = E(Var[Y|X]) + Var(E[Y|X]) = E(X^2) + Var(X) = 18 + 9 = 27$$

2. (25 points) A Markov chain with state space $\{1, 2, 3, 4, 5, 6\}$ has probability transition matrix $P$ of the following form:

$$\begin{pmatrix} 1/3 & 0 & 2/3 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 3/4 & 0 & 0 \\ 2/3 & 0 & 1/3 & 0 & 0 & 0 \\ 0 & 1/5 & 0 & 4/5 & 0 & 0 \\ 1/4 & 1/4 & 0 & 0 & 1/4 & 1/4 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \end{pmatrix}$$

Find all communication classes, which classes are transient/recurrent, and whether each class is periodic/aperiodc.

*solution:*
$1 \to 1, 3$
$2 \to 2, 4$
$3 \to 1, 3$
$4 \to 2, 4$
$5 \to 1, 2, 5, 6$
$6 \to 1, 2, 3, 4, 5, 6$
{1,3} and {2,4} are recurrent classes, while {5,6} is a transient class.
All classes are aperiodic since $P_{ii} > 0\ \forall i$.

3. (25 points) Recall the Markov chain we discussed in class concerning how many machines are broken. The state space is $\{1, 2, 3\}$. The probability transition matrix is

$$\begin{pmatrix} 0.95 & 0.05 & 0 \\ 0 & 0.9 & 0.1 \\ 1 & 0 & 0 \end{pmatrix}$$

Further assume that the probability mass function for $X_0$ is given as (0.25 0.25 0.5). Find $EX_2$.

*solution:*

$$(0.25\ 0.25\ 0.5)P^2 = (0.726\ 0.203\ 0.0025)$$

$$\begin{aligned} E(X_2) &= 1 \times P(X_2 = 1) + 2 \times P(X_2 = 2) + 3 \times P(X_2 = 3) \\ &= 1 \times 0.726 + 2 \times 0.203 + 3 \times 0.0025 = 1.139 \end{aligned}$$

4. (25 points) An urn initially contains four balls that can be green or red. A ball is drawn from the urn and replaced with a ball of the opposite color. (e.g. If red is drawn, then green is returned to the urn.) If we denote the number of red balls in the urn as $X_n$ for draw $n$, then describe the transition probabilities for this model. In other words, write down probability transition matrix.

*solution:* The state space of the Markov chain is $\{0, 1, 2, 3, 4\}$. The probability transition matrix is

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1/4 & 0 & 3/4 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 3/4 & 0 & 1/4 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

If no red balls are in the urn, we always replace a green with a red.

If one red ball and three green balls are in the urn, we replace a red ball with probability 1/4 and a green ball with probability 3/4.

If the urn has two red and green balls each, we replace a red ball with probability 1/2 and a green ball with probability 1/2.

If three red balls and one green ball are in the urn, we replace a red ball with probability 3/4 and a green ball with probability 1/4.

If the urn has four red balls, we always replace a red with a green.