Math/Stat 416 Quiz 3 Name: ____________________

Instructions: Please read!

1. Do all work on this quiz packet. It is okay to leave your answer unsimplified, as in $0.56\frac{15!}{6!9!}$ or $14e - 20$. This means that no calculator is needed.
2. Show all work for full credit! Small mistakes in arithmetic will not reduce credit if you show work; conversely, even a correct answer could get no credit without supporting work.
3. I will award partial credit where appropriate.

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1. (25 points) In a good weather year, the number of storms is Poisson distributed with mean 1; in a bad year, it is Poisson distributed with mean 3. Suppose that any year's weather conditions depends on past years on through the previous year's conditions. Suppose that a good year is equally likely to be followed by either a good or bad year, and that a bad year is twice as likely to be followed by a bad year as by a good year. Suppose that last year–call it year 0–was a good year. Find the expected total number of storms in the next two years (that is in years 1 and 2.

   solution:
   Let 0 stand for a good year and 1 for a bad year. The transition probability matrix is

$$P = \begin{pmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{pmatrix}$$

   Squaring this matrix gives

$$P^2 = \begin{pmatrix} 5/12 & 7/12 \\ 7/18 & 11/18 \end{pmatrix}$$

   If S_i is the number of storms in year i, then

$$\begin{aligned} E(S_1) &= E(S_1|X_1 = 0)P_{00} + E(S_1|X_1 = 1)P_{01} \\ &= 1 \times 1/2 + 3 \times 1/2 = 2 \\ E(S_2) &= E(S_2|X_2 = 0)P^2_{00} + E(S_2|X_2 = 1)P^2_{01} \\ &= 1 \times 5/12 + 3 \times 7/12 = 26/12 \end{aligned}$$

   and so the expected number of storms in the next two years is

$$E(S_1 + S_2) = E(S_1) + E(S_2) = 2 + 26/12 = 25/6 \approx 4.17$$

2. (25 points) The state space is $\{1, 2\}$. The probability transition matrix is

$$\begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}$$

where α and β are both strictly between 0 and 1. Does a limiting distribution exist? If so, find it.

solution:
Since this probability transition matrix defines an irreducible, ergodic Markov chain, a unique limiting distribution exists. Limiting probabilities are solutions to

$$\pi_j = \sum_i \pi_i P_{ij}$$
$$\sum_j \pi_j = 1$$

Or for this problem,

$$(1-\alpha)\pi_1 + \beta\pi_2 = \pi_1$$
$$\pi_1 + \pi_2 = 1$$

Solutions are

$$\pi_1 = \frac{\beta}{\alpha+\beta}, \quad \pi_2 = \frac{\alpha}{\alpha+\beta}$$

3. (25 points) Every time that the team wins a game, it wins its next game with probability 0.8; every time it loses a game it wins its next game with probability 0.3. Assume that the team wins the third game, what is the probability that they win the fifth game?

solution:
Let 0 stand for a win and 1 stand for a loss. The transition probability matrix is

$$P = \begin{pmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{pmatrix}$$

Squaring this matrix gives

$$P^2 = \begin{pmatrix} 0.7 & 0.3 \\ 0.45 & 0.55 \end{pmatrix}$$

Assuming the team wins the third game, the probability of winning two games later will be $P^2_{00} = 0.7$

4. (25 points) A Markov chain with state space $\{1, 2, 3, 4\}$ has probability transition matrix P of the following form:

$$\begin{pmatrix} 1/3 & 0 & 2/3 & 0 \\ 1/4 & 1/4 & 0 & 1/2 \\ 0 & 0 & 1/3 & 2/3 \\ 0 & 0 & 1/5 & 4/5 \end{pmatrix}$$

Find all communication classes, which classes are transient/recurrent, and whether each class is periodic/aperiodc. Does our theorem concerning the convergence of a Markov chain to the stationary distribution (i.e. Theorem 4.1) ensure that there is a limiting distribution? Is there a limiting distribution, i.e. does $P^n_{i,j}$ converge to some π_j regardless of i?

solution:
$\{1\}$ is transient and aperiodic.
$\{2\}$ is transient and aperiodic.
$\{3, 4\}$ is recurrent and aperiodic.
Theorem 4.1 does *not* guarantee the existence of a limiting distribution since the Markov chain is not irreducible (there is more than one communication class). However, a limiting distribution exists. The chain will leave states 1 and 2 if starting in those states and will end up in the class $\{3, 4\}$. If the transitions for only states $\{3, 4\}$ were considered as a separate Markov chain, then this Markov chain has a limiting distribution. The vector consisting of two zeros and the limiting distribution for the smaller chain would be a limiting distribution for the chain given above.