Math/Stat 416 Quiz 3 Name: ____________________

Instructions: Please read!

1. Do all work on this quiz packet. It is okay to leave your answer unsimplified, as in $0.56\frac{15!}{6!9!}$ or $14e - 20$. This means that no calculator is needed.
2. Show all work for full credit! Small mistakes in arithmetic will not reduce credit if you show work; conversely, even a correct answer could get no credit without supporting work.
3. I will award partial credit where appropriate.

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1. (25 points) Find the probability generating function of a Poisson random variable X with mean λ. (I know we've been discussing the Poisson process a great deal recently, but this problem is asking about a Poisson random variable!)

   solution:
   The probability generating function is defined as $G(z) = \sum_{k=0}^{\infty} p(k) z^k$ where for a Poisson random variable $p(k) = P(X = k) = \dfrac{e^{-\lambda}\lambda^k}{k!}$.
   Therefore,

$$\begin{aligned} G(z) &= \sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!} z^k \\ &= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda z)^k}{k!} \\ &= e^{-\lambda} e^{\lambda z} \\ &= e^{-\lambda(1-z)} \end{aligned}$$

2. Suppose that we have a branching process with a distribution for the offspring of one organism having a probability mass function

$$P(Z = 0) = \frac{1}{8} \quad P(Z = 1) = \frac{3}{8} \quad P(Z = 2) = \frac{1}{2}$$

Do the following:

(a) Find the probability generating function for Z.

(b) What is the mean of the branching process at time $n = 100$? (Without a calculator, you will not be able to give a single number as an answer. So, report an expression that would be easily calculable if you did have a calculator.)

(c) Find the probability of extinction.

solution:

(a)

$$G(z) = \sum_{k=0}^{\infty} p(k) z^k = \frac{1}{8} + \frac{3}{8} z + \frac{1}{2} z^2$$

(b)

$$\mu = \sum_{j=0}^{\infty} j P_j = \frac{3}{8} + 2\frac{1}{2} = \frac{11}{8}$$

$$E(X_{100}) = \mu^{100} = \left(\frac{11}{8}\right)^{100}$$

(c) Since $\mu > 1$, the extinction probability π_0 is the smallest posivive solution to

$$\pi_0 = \sum_{j=0}^{\infty} \pi_0^j P_j = \frac{1}{8} + \frac{3}{8}\pi_0 + \frac{1}{2}\pi_0^2$$

The solution is $\pi_0 = \frac{1}{4}$.

3. (25 points) Suppose that we have fixed times $0 < t_1 < t_2 < 2 < t_3 < 3 < t_4$. For a Poisson process with intensity λ, find the covariance of $N(t_3) - N(t_1)$ and $N(t_4) - N(t_2)$.

solution:

$\text{Cov}[N(t_3) - N(t_1), N(t_4) - N(t_2)]$

$= \text{Cov}[(N(t_3) - N(t_2)) + (N(t_2) - N(t_1)), (N(t_4) - N(t_3)) + (N(t_3) - N(t_2))]$

$= \text{Cov}[N(t_3) - N(t_2), N(t_4) - N(t_3)] +$

$\quad \text{Cov}[N(t_3) - N(t_2), N(t_3) - N(t_2)] +$

$\quad \text{Cov}[N(t_2) - N(t_1), N(t_4) - N(t_3)] +$

$\quad \text{Cov}[N(t_2) - N(t_1), N(t_3) - N(t_2)]$

$= \text{Cov}[N(t_3) - N(t_2), N(t_3) - N(t_2)]$;

the remaining terms are 0 due to independent increments

$= \text{Var}[N(t_3) - N(t_2)] = \lambda(t_3 - t_2)$

4. (25 points) Suppose that $N(t)$ is a Poisson process with intensity λ, and U is an independent uniform random variable between 0 and 1. Find the probability

$$P(N(U) = 0)$$

solution:

$$\begin{aligned} P(N(U) = 0) &= \int_0^1 P(N(U) = 0 | U = u) f(u) du \\ &= \int_0^1 \frac{(\lambda u)^0 e^{-\lambda u}}{0!} 1 du \\ &= \int_0^1 e^{-\lambda u} du \\ &= \frac{1 - e^{-\lambda}}{\lambda} \end{aligned}$$