Math/Stat 416 Quiz 5 Name: ____________________

Instructions: Please read!

1. Do all work on this quiz packet. It is okay to leave your answer unsimplified, as in $0.56\frac{15!}{6!9!}$ or $14e-20$. This means that no calculator is needed.
2. Show all work for full credit! Small mistakes in arithmetic will not reduce credit if you show work; conversely, even a correct answer could get no credit without supporting work.
3. I will award partial credit where appropriate.

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1. (25 points) $N(t)$ is a Poisson process with intensity λ. Find the covariance between $N(3)-N(1)$ and $N(4)-N(2)$.

   solution:

$$
\begin{aligned}
&\mathrm{Cov}[N(3)-N(1), N(4)-N(2)] \\
&= \mathrm{Cov}[(N(3)-N(2))+(N(2)-N(1)), (N(4)-N(3))+(N(3)-N(2))] \\
&= \mathrm{Cov}[N(3)-N(2), N(4)-N(3)] + \\
&\qquad \mathrm{Cov}[N(3)-N(2), N(3)-N(2)] + \\
&\qquad \mathrm{Cov}[N(2)-N(1), N(4)-N(3)] + \\
&\qquad \mathrm{Cov}[N(2)-N(1), N(3)-N(2)] \\
&= \mathrm{Cov}[N(3)-N(2), N(3)-N(2)]; \\
&\qquad \text{the remaining terms are 0 due to independent increments} \\
&= \mathrm{Var}[N(3)-N(2)] = \lambda(3-2) = \lambda
\end{aligned}
$$

2. (25 points) Suppose that $N(t)$ is a Poisson process with intensity λ. Find the following:

   (a) $P[N(2) = 3|N(5) = 5]$

   (b) $E[N(2)|N(5) = 5]$

solution:

(a)

$$
\begin{aligned}
P[N(2) = 3|N(5) = 5] &= \frac{P[N(5) = 5|N(2) = 3]P[N(2) = 3]}{P[N(5) = 5]} \\
&= \frac{P[N(3) = 2]P[N(2) = 3]}{P[N(5) = 5]} = \frac{\frac{e^{-3\lambda}(3\lambda)^2}{2!}\frac{e^{-2\lambda}(2\lambda)^3}{3!}}{\frac{e^{-5\lambda}(5\lambda)^5}{5!}} \\
&= \frac{5!}{3!2!}\left(\frac{2}{5}\right)^3\left(\frac{3}{5}\right)^2
\end{aligned}
$$

(b) Recognize the solution to part (a) as a Binomial probability. That is, $N(2)|N(5) = 5 \sim \text{Binomial}(5, \frac{2}{5})$.
Therefore $E[N(2)|N(5) = 5] = 5 \times \frac{2}{5} = 2$.

3. (25 points) Suppose customers arrive to a ticket window according to a Poisson process with intensity/rate of $\lambda = 7$ per hour.

   (a) What is the variance of the time when the fourth customer arrives?

   (b) Assume that 40 percent of the customers are men and 60 percent are women (and that whether each customer is a man/woman is independent of every other customer). What is the expected time when the fifth woman arrives?

solution:

(a) Denote the interarrival time of the i^{th} customer by T_i.
We know $T_i \overset{iid}{\sim} \text{Exp}(7)$. The variance of the arrival time of the fourth customer is

$$Var\left(\sum_{i=1}^{4} T_i\right) = \sum_{i=1}^{4} Var(T_i) = 4 \times \left(\frac{1}{7}\right)^2 = 0.082$$

(b) By Propoision 5.2, the number of women who arrive is a Poission process with rate $0.6\lambda = 4.2$. Let W_i denote the interarrival time of the i^{th} woman. $W_i \overset{iid}{\sim} \text{Exp}(4.2)$. The expected arrival time of the fifth woman is

$$E\left(\sum_{i=1}^{5} W_i\right) = \sum_{i=1}^{5} E(W_i) = 5 \times \left(\frac{1}{4.2}\right) = 1.19$$

4. (25 points) Suppose that $N(t)$ is a non-homogeneous Poisson process with intensity function $\lambda(t) = 3t^2 + 2t$. Find $E[N(3) - N(2)]$.

solution:

$$N(3) - N(2) \sim \text{Poisson}\left(\int_2^3 \lambda(t)dt\right)$$

$$E[N(3) - N(2)] = \int_2^3 \lambda(t)dt = 24$$