Instructions: Please read!

1. Do all work on this quiz packet. It is okay to leave your answer unsimplified, as in $0.56 \frac{151}{599}$ or $14e - 20$. This means that no calculator is needed.

2. Show all work for full credit! Small mistakes in arithmetic will not reduce credit if you show work; conversely, even a correct answer could get no credit without supporting work.

3. I will award partial credit where appropriate.

1. Suppose we have a discrete-time Markov chain with probability transition matrix

$$P = \begin{pmatrix} 0.25 & 0.25 & 0.5 \\ 0.3 & 0.6 & 0.1 \\ 0.2 & 0.5 & 0.3 \end{pmatrix}$$

and $N(t)$ is a Poisson process with intensity 4. Find the distribution of the continuous time Markov chain $X_{N(t)}$ by

(a) writing down the $Q$ matrix,

(b) and also writing down the $v_i$ and $P^*_ij$ as we defined them in class.

solution:

(a) Off-diagonal elements $Q_{ij} = 4P_{ij}$, and $Q_{ii} = -\sum_{j\neq i} Q_{ij}$.

$$Q = \begin{pmatrix} -3 & 1 & 2 \\ 1.2 & -1.6 & 0.4 \\ 0.8 & 2 & -2.8 \end{pmatrix}$$

(b) $v_i = -Q_{ii}$; $v_0 = 3, v_1 = 1.6, v_2 = 2.8$.

$P^*_{ij} = \frac{q_{ij}}{v_i}$;

$P^*_{01} = 1/3, P^*_{02} = 2/3$

$P^*_{10} = 1/4, P^*_{12} = 3/4$

$P^*_{20} = 2/7, P^*_{21} = 5/7$
2. Define the continuous time Markov chain $X(t)$ which takes values 0 or 1 by rate matrix

$$Q = \begin{pmatrix} -3 & 3 \\ 4 & -4 \end{pmatrix}$$

Suppose that the chain starts in state 0 at time $t = 0$. Define $T$ as the time for the chain to leave state 0, move to state 1, and then return to state 0.

(a) Find the expected value of $T$.
(b) Find the variance of $T$.

**solution:**

$T_{01} \sim \text{Exp}(3), \; T_{10} \sim \text{Exp}(4). \; T = T_{01} + T_{10}$.

(a) $E(T) = E(T_{01}) + E(T_{10}) = \frac{1}{3} + \frac{1}{4}$

(b) $\text{Var}(T) = \text{Var}(T_{01}) + \text{Var}(T_{10}) = \left(\frac{1}{3}\right)^2 + \left(\frac{1}{4}\right)^2$
3. A small barbershop, operated by a single barber, has room for at most two customers. Potential customers arrive at a Poisson rate of four per hour, and the successive service times are independent exponential random variables with mean $1/2$ hour.

(a) Write down the rate matrix $Q$ for this model.

(b) Write down the holding rates ($v_i$) and the transitioning probabilities given that you leave a state ($P_{ij}^*$) for this model.

**solution:**

(a) This is a birth-death process with $\lambda_0 = \lambda_1 = 4$, $\mu_1 = \mu_2 = 2$. The rate matrix is therefore

$$Q = \begin{pmatrix} -4 & 4 & 0 \\ 2 & -6 & 4 \\ 0 & 2 & -2 \end{pmatrix}$$

(b) $v_i = -Q_{ii}; v_0 = 4, v_1 = 6, v_2 = 2$.

- $P_{ij}^* = \frac{q_{ij}}{v_i};$
- $P_{01}^* = 1, P_{02}^* = 0$
- $P_{21}^* = 1, P_{20}^* = 0$
- $P_{10}^* = 2/6, P_{12}^* = 4/6$
4. Suppose that \( N(t) \) is a Poisson process with intensity \( \lambda \) and \( Z \) is an exponential random variable with intensity \( \mu \). In other words, the density of \( Z \) is
\[
f(z) = \mu e^{-\mu z}
\]
Find
\[
P(N(Z) = 0)
\]
(Note: This is a problem from the Poisson process portion of the course and does not involve continuous time Markov chains.)

**solution:**
\[
P(N(Z) = 0) = \int_0^1 P(N(Z) = 0 | Z = z) f(z) dz
\]
\[
= \int_0^\infty \frac{(\lambda z)^0 e^{-\lambda z}}{0!} \mu e^{-\mu z} dz
\]
\[
= \int_0^\infty \mu e^{-(\lambda + \mu) z} dz
\]
\[
= \mu \frac{-1}{\lambda + \mu} e^{-(\lambda + \mu) z} \bigg|_{z=0}^\infty
\]
\[
= \frac{\mu}{\lambda + \mu}
\]