Math/Stat 416 Quiz 6 Name: ____________________

Instructions: Please read!

1. Do all work on this quiz packet. It is okay to leave your answer unsimplified, as in $0.56\frac{15!}{6!9!}$ or $14e - 20$. This means that no calculator is needed.
2. Show all work for full credit! Small mistakes in arithmetic will not reduce credit if you show work; conversely, even a correct answer could get no credit without supporting work.
3. I will award partial credit where appropriate.

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1. Suppose we have a discrete-time Markov chain with probability transition matrix

$$P = \begin{pmatrix} 0.25 & 0.25 & 0.5 \\ 0.3 & 0.6 & 0.1 \\ 0.2 & 0.5 & 0.3 \end{pmatrix}$$

and $N(t)$ is a Poisson process with intensity 4. Find the distribution of the continuous time Markov chain $X_{N(t)}$ by

(a) writing down the Q matrix,

(b) and also writing down the v_i and P^*_{ij} as we defined them in class.

solution:

(a) Off-diagonal elements $Q_{ij} = 4P_{ij}$, and $Q_{ii} = -\sum_{j\neq i} Q_{ij}$.

$$Q = \begin{pmatrix} -3 & 1 & 2 \\ 1.2 & -1.6 & 0.4 \\ 0.8 & 2 & -2.8 \end{pmatrix}$$

(b) $v_i = -Q_{ii}$; $v_0 = 3, v_1 = 1.6, v_2 = 2.8$.
$P^*_{ij} = \frac{q_{ij}}{v_i}$;
$P^*_{01} = 1/3, P^*_{02} = 2/3$
$P^*_{10} = 1/4, P^*_{12} = 3/4$
$P^*_{20} = 2/7, P^*_{21} = 5/7$

2. Define the continuous time Markov chain $X(t)$ which takes values 0 or 1 by rate matrix

$$Q = \begin{pmatrix} -3 & 3 \\ 4 & -4 \end{pmatrix}$$

Suppose that the chain starts in state 0 at time $t = 0$. Define T as the time for the chain to leave state 0, move to state 1, and then return to state 0.

(a) Find the expected value of T.

(b) Find the variance of T.

solution:

$T_{01} \sim \mathrm{Exp}(3)$, $T_{10} \sim \mathrm{Exp}(4)$. $T = T_{01} + T_{10}$.

(a) $E(T) = E(T_{01}) + E(T_{10}) = \frac{1}{3} + \frac{1}{4}$

(b) $Var(T) = Var(T_{01}) + Var(T_{10}) = \left(\frac{1}{3}\right)^2 + \left(\frac{1}{4}\right)^2$

3. A small barbershop, operated by a single barber, has room for at most two customers. Potential customers arrive at a Poisson rate of four per hour, and the successive service times are independent exponential random variables with mean 1/2 hour.

   (a) Write down the rate matrix Q for this model.

   (b) Write down the holding rates (v_i) and the transitioning probabilities given that you leave a state (P^*_{ij}) for this model.

   solution:

   (a) This is a birth-death process with $\lambda_0 = \lambda_1 = 4$, $\mu_1 = \mu_2 = 2$. The rate matrix is therefore

   $$Q = \begin{pmatrix} -4 & 4 & 0 \\ 2 & -6 & 4 \\ 0 & 2 & -2 \end{pmatrix}$$

   (b) $v_i = -Q_{ii}$; $v_0 = 4, v_1 = 6, v_2 = 2$.
   $P^*_{ij} = \frac{q_{ij}}{v_i}$;
   $P^*_{01} = 1, P^*_{02} = 0$
   $P^*_{21} = 1, P^*_{20} = 0$
   $P^*_{10} = 2/6, P^*_{12} = 4/6$

4. Suppose that $N(t)$ is a Poisson process with intensity λ and Z is an exponential random variable with intensity μ. In other words, the density of Z is

$$f(z) = \mu e^{-\mu z}$$

Find

$$P(N(Z) = 0)$$

(Note: This is a problem from the Poisson process portion of the course and does not involve continuous time Markov chains.)

solution:

$$\begin{aligned}
P(N(Z) = 0) &= \int_0^1 P(N(Z) = 0 | Z = z) f(z) dz \\
&= \int_0^\infty \frac{(\lambda z)^0 e^{-\lambda z}}{0!} \mu e^{-\mu z} dz \\
&= \int_0^\infty \mu e^{-(\lambda+\mu)z} dz \\
&= \mu \frac{-1}{\lambda + \mu} e^{-(\lambda+\mu)z} |_{z=0}^{\infty} \\
&= \frac{\mu}{\lambda + \mu}
\end{aligned}$$