Homework 10: solutions

STAT 414 - Spring 2015

1) a) $E(XY) = \int_0^1 \int_0^y xy \frac{1}{y} \, dx \, dy = \int_0^1 \frac{y^2}{2} dy = \frac{1}{6}$ //

b) $E(X) = \int_0^1 \int_0^y x \frac{1}{y} \, dx dy = \int_0^1 \frac{y}{2} dy = \frac{1}{4}$ //

c) $E(Y) = \int_0^1 \int_0^y y \frac{1}{y} dy = \int_0^1 y \, dy = \frac{1}{2}$ //

In parts b and c, we have combined the steps of calculating marginal density and evaluating expectation.

2) For a single rolle of a fair die with outcome X_i,

$E(X_i) = 3.5$, $V(X_i) =$

We can assume that the outcome of each roll is independent

$\therefore E\left(\sum_{i=1}^{10} X_i\right) = \sum_{i=1}^{10} E(X_i) = 10 \times 3.5 = 35$ //

$V\left(\sum_{i=1}^{10} X_i\right) = \sum_{i=1} V(X_i) = 10 \times \frac{35}{12} = 29.17$ //   Only because of independence

3) $Z \sim N(0,1)$ and $X = Z$ if $z > x$
$= 0$ otherwise

$$\therefore E(X) = \int_{y > x} y \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy$$

$$= \frac{e^{-x^2/2}}{\sqrt{2\pi}}$$ //

4) X and Y are iid with $E(X) = E(Y) = \mu$
$V(X) = V(Y) = \sigma^2$

$E[(X-Y)^2] = Var(X-Y)$ Since $E(X-Y) = E(X) - E(Y) = 0$
$= Var(X) + var(Y)$ Since $X \perp\!\!\!\perp Y$
$= 2\sigma^2$ //

5) $E(X) = 1$, $V(X) = 5$

a) $E[(2+X)^2] = E(4 + 4X + X^2)$
$= E(4) + 4E(X) + E(X^2)$
$= E(4) + 4E(X) + V(X) + [E(X)]^2$ $V(X) = E(X^2) - [E(X)]^2$
$= 4 + 4 + 5 + 1$
$= \cancel{12}$ // 14 //

b) $V(4+3X) = V(3X) = 9V(X) = 45$ //

6) $f(x,y) = \frac{2e^{-2x}}{x} I\{0<y<x<\infty\}$

i) $E(XY) = \int_0^\infty \int_0^x xy \frac{2e^{-2x}}{x} dy\, dx = \int_0^\infty x^2 e^{-2x} dx = \frac{1}{8}\int_0^\infty y^2 e^{-y} dy$

$E(XY) = \frac{1}{4}$ //

ii) $E(X) = \int_0^\infty x \int_0^x \frac{2e^{-2x}}{x} dy\, dx = \int_0^\infty x \cdot 2e^{-2x} dx$

$E(X) = \frac{1}{2}$ //

iii) $E(Y) = \int_0^\infty y \int_0^\infty \frac{2e^{-2x}}{x} dx\, dy$

$= \int_0^\infty \int_0^x y \frac{2e^{-2x}}{x} dy\, dx = \int_0^\infty x e^{-2x} dx = \frac{1}{4}\int y e^{-2y} dy = \frac{1}{4}$ //

$\therefore \text{cov}(X,Y) = E(XY) - E(X)E(Y)$

$= \frac{1}{4} - \frac{1}{2}\cdot\frac{1}{4}$

$\text{cov}(X,Y) = \frac{1}{8}$ //

7) $X_1, X_2 \ldots$ are independent with $E(X_i) = \mu$, $V(X_i) = \sigma^2$

$Y_n = X_n + X_{n+1} + X_{n+2}$

For $j \geq 0$,

- $\text{cov}(Y_n, Y_{n+0}) = \text{var}(Y_n) = V(X_n + X_{n+1} + X_{n+2}) = 3\sigma^2$ //
- $\text{cov}(Y_n, Y_{n+1}) = \text{cov}(X_n + X_{n+1} + X_{n+2}, X_{n+1}, X_{n+2}, X_{n+3})$

$= \text{cov}(X_{n+1} + X_{n+2}, X_{n+1}, X_{n+2})$ since $X_n \perp\!\!\!\perp X_{n+3}$

$= \text{var}(X_{n+1}, X_{n+2})$

$= 2\sigma^2$ //

- $\text{cov}(Y_n, Y_{n+2}) = \text{cov}(X_{n+2}, X_{n+2})$ Similar to above

$= \sigma^2$ //

- $\text{cov}(Y_n, Y_{n+j}) = 0 \quad \forall j \geq 3$

8) $f(x,y) = \dfrac{e^{-x/y} e^{-y}}{y} I\{x>0, y>0\}$

$$f_{X|Y}(x|y) = \frac{e^{-x/y} e^{-y}/y}{\int_0^\infty e^{-x/y} e^{-y}/y \, dx} = \frac{1}{y} e^{-x/y} \qquad 0 < x < y$$

$\therefore$ Given $Y = y$, $X \sim$ exponential (mean $= y$)

$\therefore E(X^2 / Y = y) = 2y^2$ //

9) Let X : # storms in a year and G (B) are ~~$X \sim Poi(3)$~~ events defining good (bad) year

$$E(X) = E(X/G)\,P(G) + E(X|B)\,P(B)$$
$$= 3(0.4) + 5(0.6)$$
$$= \underline{\underline{4.2}}$$

Since $E(Z^2) = \lambda + \lambda^2$ for $Z \sim Poi(\lambda)$

$$\therefore E(X^2) = E(X^2/G)\,P(G) + E(X^2/B)\,P(B)$$
$$= 12(0.4) + 30(0.6) = 22.8$$

$$\therefore V(X) = 22.8 - (4.2)^2 = 5.16 \;//$$

10) Let X : # accidents in a year
Let Y be the event indicating the sub-population that a sample belongs to

~~$E(X^2) = \frac{1}{3}\{$~~

a) $P(0 \text{ accidents}) = 0.6e^{-2} + 0.4e^{-3}$

b) $P(3 \text{ accidents}) = 0.6e^{-2}\cdot\frac{2^3}{3!} + 0.4e^{-3}\frac{3^3}{3!}$

c) $$P(3/0) = \frac{P(3,0)}{P(0)} = \frac{0.6e^{-2}e^{-2}\frac{2^3}{3!} + 0.4e^{-3}e^{-3}\frac{3^3}{3!}}{0.6e^{-2} + 0.4e^{-3}}$$

11) $X_1 \sim \exp(\lambda_1)$, $X_2 \sim \exp(\lambda_2)$

$\therefore M_{X_1}(t) = \dfrac{\lambda_1}{\lambda_1 - t}$ and $M_{X_2}(t) = \dfrac{\lambda_2}{\lambda_2 - t}$

Since $X_1 \perp\!\!\!\perp X_2$, $M_{X_1+X_2}(t) = M_{X_1}(t) \cdot M_{X_2}(t)$

But we have $Y = X_1 - X_2$

$X_1 \perp\!\!\!\perp X_2 \implies X_1 \perp\!\!\!\perp -X_2$

$M_{-X_2}(t) = \cancel{Bx}\ M_{X_2}(-t)$ By property

$\implies M_{-X_2}(t) = \dfrac{\lambda_2}{\lambda_2 + t}$

$$\begin{aligned}
\therefore M_Y(t) &= M_{X_1 - \lambda_2}(t) \\
&= M_{X_1}(t) \cdot M_{-X_2}(t) \\
&= \frac{\lambda_1}{\lambda_1 - t} \cdot \frac{\lambda_2}{\lambda_2 + t} \quad //
\end{aligned}$$

12) $f(x, \cancel{y}) = \left(\frac{1}{2}x + \frac{1}{2}\right) I\{-1 < x < 1\}$

$$\begin{aligned}
M_X(t) &= \int_{-1}^{1} \frac{e^{tx}}{2} x \, dx + \int_{-1}^{1} \frac{e^{tx}}{2} dx \\
&= \left[\frac{x e^{tx}}{2t}\right]_{-1}^{1} - \left[\frac{e^{tx}}{2t^2}\right]_{-1}^{1} + \left[\frac{e^{tx}}{2t}\right]_{-1}^{1} \\
&= \frac{e^t}{2t} + \cancel{\frac{e^{-t}}{2t}} - \frac{e^t}{2t^2} + \frac{e^{-t}}{2t^2} + \frac{e^t}{2t} - \cancel{\frac{e^{-t}}{2t}} \\
&= e^t\left[\frac{1}{2t} - \frac{1}{2t^2} + \frac{1}{2t}\right] + \frac{e^{-t}}{2t^2}
\end{aligned}$$

$$M_X(t) = e^t\left[\frac{1}{t} - \frac{1}{2t^2}\right] + \frac{e^{-t}}{2t^2} \quad //$$