# HOMEWORK 3: Solutions

## STAT 414: Spring 2015

1)

| $i$ | P(Total = $i$) | P(at least 1 six & total = $i$) | P(at least 1 six / total = $i$) |
|---|---|---|---|
| 7 | 6/36 | 2/36 | 2/6 |
| 8 | 5/36 | 2/36 | 2/5 |
| 9 | 4/36 | 2/36 | 2/4 |
| 10 | 3/36 | 2/36 | 2/3 |
| 11 | 2/36 | 2/36 | 2/2 |
| 12 | 2/36 | 2/36 | 2/2 |

For totals $i = 2, 3 \ldots 6$, it is not possible to have a six on one of the dice and get the total. Hence those probabilities are zero.

2) To find: P(1st & 3rd balls white / exactly 3 balls are white), where sample size = 4 and we have 12 balls (8 white)

<u>With replacement:</u>

#(1st & 3rd white AND exactly 3 are white):

$\underline{W}\ \ \frac{W}{NW}\ \ \underline{W}\ \ \frac{NW}{W} \Rightarrow 8 \times 8 \times 8 \times 4 \times 2$ ways, since 2nd and 4th position can be occupied in 2 ways

#(Exactly 3 balls white):

$8 \times 8 \times 8 \times \cancel{8} \times 4 \times {}^4C_3$ ways, since the 3 positions for white balls can be chosen in ${}^4C_3$ ways

$\therefore$ Required probability = $\boxed{\frac{1}{2}}$

Without replacement:

#( $1^{st}$ and $3^{rd}$ white AND exactly three white):

$8 \times 7 \times 6 \times 4 \times 2$ , since there are again 2 ways to position a white and non-white ball in $2^{nd}$-$4^{th}$ place

Since we are not putting a chosen ball back, we have 1 less white ball for every choice

# (Exactly three white balls):

$8 \times 7 \times 6 \times 4 \times {}^4C_3$ , similar to the reasoning in earlier part

$\therefore$ Required probability = $\boxed{\frac{1}{2}}$

☆ In this answer, many students used the 'n choose k' formula directly. Although this part of the problem does require selection without replacement, and 'n choose k' selects in the same way; we are constrained by the positioning of white balls ($1^{st}$ and $3^{rd}$). Therefore this approach would not yield correct answer in this case.

3) P($1^{st}$ draw spade / $2^{nd}$ and $3^{rd}$ draws are spades)

$$= \frac{P(1^{st} \text{ spade} \cap 2^{nd} \, 3^{rd} \text{ spades})}{P(2^{nd} \, \& \, 3^{rd} \text{ spades})} = \frac{P(1^{st} \text{ 3 draws spades})}{P(2^{nd}\text{-}3^{rd} \text{ spades})}$$

In the denominator, we have 2 cases. The $1^{st}$ draw can be a spade or not a spade. Since we are drawing without replacement, one card reduces every time.

$$\therefore \text{Required Probability} = \frac{13/52 \times 12/51 \times 11/50}{(13/52 \times 12/51 \times 11/50) + (39/52 \times 13/51 \times 12/50)} = \boxed{\frac{11}{50}} \approx 0.22$$

4) $P(E_1) = P($1st hand has exactly 1 ace$)$

$$= \frac{{}^4C_1 \, {}^{48}C_{12}}{{}^{52}C_{13}}$$

Since we are choosing from the entire deck

Now that we know the first ace is gone into the first stack and so are 13 cards already gone,

$P(E_2|E_1) = P($2nd hand has exactly 1 ace, given that 1st hand also had exactly 1 ace$)$

$$= \frac{{}^3C_1 \, {}^{36}C_{12}}{{}^{39}C_{13}}$$

Similarly,

$$P(E_3|E_1, E_2) = \frac{{}^2C_1 \, {}^{24}C_{12}}{{}^{26}C_{13}}$$

Finally, since there are only 13 cards left and only one ace among them,

$$P(E_4/E_1, E_2, E_3) = \frac{{}^1C_1 \, {}^{12}C_{12}}{{}^{13}C_{13}} = 1$$

$$\therefore P(E_1, E_2, E_3, E_4) = \frac{{}^4C_1 \, {}^{48}C_{12}}{{}^{52}C_{13}} \times \frac{{}^3C_1 \, {}^{36}C_{12}}{{}^{39}C_{13}} \times \frac{{}^2C_1 \, {}^{24}C_{12}}{{}^{26}C_{13}} \approx 0.105$$

(This answer matches with example 2g, chapter 3, in textbook)

5) Let $\underline{S}$: Event that a baby survives, $\underline{C}$: Event that delivery was cesarean

Given: $P(S) = 0.98$, $P(C) = 0.15$, $P(S|C) = 0.96$

$$\therefore P(S|C^c) = \frac{P(S \cap C^c)}{P(C^c)} = \frac{P(S) - P(S \cap C)}{1 - P(C^c)}$$ → Bayes' formula

$$= \frac{P(S) - [P(S|C) \times P(C)]}{1 - P(C)} = \boxed{0.9835}$$

6) $P(\text{No white}) = \frac{7}{12} \times \frac{8}{13} \times \frac{9}{14} = \underline{\underline{\frac{3}{13}}}$

$P(1\text{ white}) = \left(\frac{5}{12} \times \frac{7}{13} \times \frac{8}{14}\right) \times 3 = \underline{\underline{\frac{5}{13}}}$ Since 3 positions at which white ball can be placed

$P(2\text{ whites}) = \left(\frac{5}{12} \times \frac{6}{13} \times \frac{7}{14}\right) \times 3 = \underline{\underline{\frac{15}{52}}}$ Since red ball can be placed in 3 positions

$P(3\text{ whites}) = \frac{5}{12} \times \frac{6}{13} \times \frac{7}{14} = \underline{\underline{\frac{5}{52}}}$

☆ In this problem, key is to remember that we are adding a ball every time and that the draws are sequential.

~~7) We have to find:~~
~~P(the coin flipped to head was 5th coin~~

7) We have to find:

$P(5^{th}\text{ coin was drawn} / \text{it showed head}) = \dfrac{P(5^{th}\text{ coin will show a head})}{P(\text{seeing a head})}$

$$= \frac{5/10}{\sum_{i=1}^{10} i/10} = 0.0909$$

The denominator is calculated using an extension of Bayes' formula; probability of seeing a head is equal to a summation of probabilities of seeing a head under all possible conditions

8) Let <u>Acc</u> denote accident, $A \equiv$ average, $g \equiv$ good and $B \equiv$ bad
$P(Acc/g) = 0.05$, $P(Acc/A) = 0.15$, $P(Acc/B) = 0.3$, $P(g) = 0.2$, $P(A) = 0.5$, $P(B) = 0.3$

- $P(Acc) = P(Acc \cap A) + P(Acc \cap g) + P(Acc \cap B)$ Bayes' rule

$= P(Acc/A)\,P(A) + P(Acc/g)P(g) + P(Acc/B)P(B)$

$= 0.175$

$\therefore$ 17.5% of the policyholders have an accident in a fixed year

- $$P(g/Acc^c) = \frac{P(g \cap Acc^c)}{P(Acc^c)} = \frac{P(Acc^c/g)\,P(g)}{P(Acc^c)}$$

$$= \frac{[1-P(Acc/g)]\,P(g)}{1-P(Acc)}$$ Rules of conditional probability

$$= \boxed{0.2303}$$

- Similarly as above, $P(A/Acc^c) = \boxed{0.5151}$

☆ Many of you interpreted the last question to mean that $P(g \cup A / Acc^c)$ needs to be calculated. If that numerical answer is correct, no points have been deducted.

g)

| | G | B | |
|---|---|---|---|
| First | 6 | 4 | 10 |
| Sophomore | k | 6 | 6+k |
| | 6+k | 10 | $\boxed{16+k}$ |

We know that, for gender and year to be independent, each of the 4 combination of events must be independent. Solving for,

$P(Boy) \cdot P(\text{First year}) = P(Boy \cap \text{First year})$

i.e. $$\frac{10}{16+k} \cdot \frac{10}{16+k} = \frac{4}{16+k}$$

gives us $\boxed{k=9}$

We can verify that this tells us, each of the 4 are independent $\Longrightarrow$ gender and year are independent

10) The experiment concludes when the most recent pair of flips results in a (H,T) or (T, H)

a) ∴ P(outcome is H) = P(T,H / flips are (H,T) or (T,H))

$$= \frac{P(T,H)}{P(T,H) + P(H,T)}$$

$$= \frac{p(1-p)}{p(1-p)+(1-p)p} = \boxed{\frac{1}{2}}$$

Similarly P(outcome is T) $= \boxed{\frac{1}{2}}$

This makes intuitive sense because among the last two flips, the orderings (H,T) and (T,H) are equally likely [p(1-p)]. And only one of these two give us the required outcome (of heads or tails).

∴ It is equally likely that the outcome will be a head/tail.

b) If we flip continuously until the last two flips are different, the outcome depends on all the previous tosses

i.e. P(head is outcome) = P(all perious outcomes were tails)

Say the experiment terminates at $k^{th}$ trial ;

$P(head) = (1-p)^{k-1} \cdot p$ For (k-1) tails and 1 head

Similarly,

$P(tail) = p^{k-1}(1-p)$

∴ This procedure is not giving the same result as the earlier one, EXCEPT when $p = (1-p) = \frac{1}{2}$