HOMEWORK 5: Solutions

STAT 414 : Spring 2015

1) For $X \sim \text{Poisson}(\lambda)$, $P(X=x) = \dfrac{e^{-\lambda}\lambda^x}{x!}$

$$E[X(X-1)] = \sum_{x=0}^{\infty} x(x-1)\frac{e^{-\lambda}\lambda^x}{x!}$$
$$= \sum_{x=0}^{\infty} (x^2-x)\frac{e^{-\lambda}\lambda^x}{x!}$$
$$= \sum_{x=0}^{\infty} x^2\frac{e^{-\lambda}\lambda^x}{x!} - \sum_{x=0}^{\infty} x\frac{e^{-\lambda}\lambda^x}{x!}$$
$$= \lambda(\lambda+1) - \lambda$$

Refer to Example 7c (Chapter 4)

$$= \underline{\underline{\lambda^2}}$$

2) If the newsboy purchases 'a' (say) newspapers,

$$\text{Profit} = \begin{cases} -10a + 15X & \text{when } X < a \\ 5a & \text{when } X \geq a \end{cases}$$

where Demand : $X \sim \text{Binomial}(10, 1/3)$

Solving for some initial values tells us that the profit is maximum at $a = 3$.

We need not check for higher values as it is easy to see that $E(\text{demand}) = 3.33 \Rightarrow$ probability of higher values is lower. Therefore he will begin to go into losses very quickly.

3) It is established that sampling is with replacement

$\therefore$ Draws are independent and probability of success (white ball) is the same at each draw.

$\therefore$ Let X: Number of white balls in 4 draws

$X \sim \text{Binomial}(4, 1/2)$

$$\therefore P(X=2) = {}^4C_2\left(\tfrac{1}{2}\right)^2\left(\tfrac{1}{2}\right)^{4-2} = \frac{3}{8} = 0.375$$

4) Let X : Number of heads (tails) in 10 tosses of a fair coin

$X \sim \text{Binomial}(10, \tfrac{1}{2})$

If the man did not have ESP, he would have rightly predicted the outcome as per the distribution above

$\therefore P(\text{Doing at least as well as 7 correct guesses}) = \sum_{i=7}^{10} {}^{10}C_i \left(\frac{1}{2}\right)^i \left(\frac{1}{2}\right)^{10-i}$

$= 0.1719$

5) $P(\text{Prosecution}/\text{guilt}) = 0.7$

$P(\text{Prosecution}/\text{innocence}) = 0.3$

Since each judge votes independently and with same probabilities,

$(\text{Decision}|\text{guilt})X \sim \text{Binomial}(n = \#\text{judges},\ p = \text{Prob. of prosecution})$

where X is defined as the decision, given the guilt (or not)

For 7 judges, majority would be 4 or more judges and for 8 & 9 judges, it would be 5 or more

a) Let g: guilty and DG: declaring guilty

$P(DG/g)_{9 \text{ judges}} = \sum_{i=5}^{9} {}^{9}C_i\, 0.7^i\, 0.3^{9-i} = \underline{0.901}$

$P(DG/g)_{8 \text{ judges}} = \sum_{i=5}^{8} {}^{8}C_i\, 0.7^i\, 0.3^{8-i} = \underline{0.806}$

$P(DG/g)_{7 \text{ judges}} = \sum_{i=4}^{7} {}^{7}C_i\, 0.7^i\, 0.3^{7-i} = \underline{0.874}$

b) $P(DG/g^c)_{9 \text{ judges}} = \sum_{i=5}^{9} {}^{9}C_i\, 0.3^i\, 0.7^{9-i} = \underline{0.099}$

$P(DG/g^c)_{8 \text{ judges}} = \sum_{i=5}^{8} {}^{8}C_i\, 0.3^i\, 0.7^{8-i} = \underline{0.058}$

$P(DG/g^c)_{7 \text{ judges}} = \sum_{i=4}^{7} {}^{7}C_i\, 0.3^i\, 0.7^{7-i} = \underline{0.126}$

c) Underlying assumption is that a defence attorney is trying to set the accused free

$\therefore$ He will check the probability of prosecution and minize it (or maximize the probability of not prosecution)

$$P(\text{Prosecution}) = P(\text{Prosecution} \cap \text{guilty}) + P(\text{prosecution} \cap \text{guilty}^c)$$
$$= \cancel{P(\text{Prose}}$$
$$= P(DG/g)\,P(g) + P(DG/g^c)\,P(g^c)$$

We know that $P(g) = 0.6$, $P(g^c) = 0.4$

$\therefore$ Using parts a & b, we calculate,

For 9 judges : $P(DG) = \cancel{(0.6 \times 0.901}$
$= (0.6 \times 0.901) + (0.4 \times 0.099) = 0.5802$

For 8 judges : $P(DG) = (0.6 \times 0.806) + (0.4 \times 0.058) = 0.5068$

For 7 judges : $P(DG) = (0.6 \times 0.874) + (0.4 \times 0.126) = 0.5748$

$\therefore$ Defence attorney should make 1 preemptory challenge

6) a) $P(htt \text{ in first } 3 / 6 \text{ heads})$

$$= \frac{P(htt \text{ first three outcomes } \& \text{ } 6 \text{ heads})}{P(6 \text{ heads})}$$

$$= \frac{p(1-p)^2 \cdot {}^7C_5\, p^5 (1-p)^2}{{}^{10}C_6\, p^6 (1-p)^4}$$

$$= \frac{1}{10}$$

b) Same as part (a) since the position of head does not change probabilities.

7) We know that typographical errors can be modeled as a poisson random variable

∴ Let X : Typographical errors on a given page

$X \sim Poisson\ (0.2)$

a) $P(X=0) = \dfrac{e^{-0.2}\, 0.2^0}{0!} = e^{-0.2}$

b) $P(X \geq 2) = 1 - P(X < 2)$

$= 1 - P(X=0) - P(X=1)$

$= 1 - \dfrac{e^{-0.2}\, 0.2^0}{0!} - \dfrac{e^{-0.2}\, 0.2^1}{1!}$

$= 1 - 1.2\, e^{-0.2}$

8) a) Since a date is fixed,

P(two people are independently born on the same day) $= \dfrac{1}{365} \times \dfrac{1}{365}$

we are considering 80,000 pairs of people

∴ we see that 'n' is large and 'p' small

∴ X : # couples born on the same day $\sim$ Poisson$(n \times p)$

i.e. $X \sim Poisson\left(80000 \times \dfrac{1}{365} \times \dfrac{1}{365}\right)$

$\therefore P(X \geq 1) = 1 - P(X=0)$

$= \dfrac{1 - e^{-\frac{80000}{365^2}} \left(\frac{80000}{365^2}\right)^0}{0!}$

≈ 0.4518

b) Similarly, the X here is ; $X \sim Poisson\left(\dfrac{80000}{365}\right)$

$\therefore P(X \geq 1) \approx 1$

This is not surprising, since as we learnt in example 5i of chapter 2, for 50 people in a room, atleast 2 share the same birthday is 0.97

Assumptions:

We make some important assumptions in calculating these answers

1) The year is not leap
2) The probability of each couple sharing birthdate is identical
3) Each couple may be born on the same day independently of other couples

9) We want to calculate $P(\text{Drug beneficial} / 2 \text{ colds})$

ie. $P(\lambda = 3 / X = \overset{2}{\cancel{5}})$ where $X \sim \text{Poisson}(\lambda = 5)$ & $\lambda = 3$

$$P(\lambda = 3 / X = 5) = \frac{P(\lambda = 3, X = 5)}{P(X = 5)}$$

$$= \frac{P(\lambda = 3, X = 5)}{P(X = 5, \lambda = 3) + P(\lambda}$$

$$P(\lambda = 3 / X = 2) = \frac{P(\lambda = 3, X = 2)}{P(X = 2)}$$

$$= \frac{P(\lambda = 3, X = 2)}{P(\lambda = 3, X = 2) + P(\lambda = 5, X = 2)}$$

$$= \frac{P(X = 2 / \lambda = 3)\, P(\lambda = 3)}{P(X = 2 / \lambda = 3)\, P(\lambda = 3) + P(X = 2 / \lambda = 5)\, P(\lambda = 5)}$$

$$= \frac{\frac{e^{-3} 3^2}{2!} \cdot 0.75}{\frac{e^{-3} 3^2}{2!} 0.75 + \frac{e^{-5} 5^2}{2!} 0.25}$$

$P(\lambda = 3 / X = 2) = 0.8886$

$\therefore$ There is 0.8886 chance that the drug is beneficial for this person, even if he contracted 2 colds

10) Assuming independently dealt hands and equal probability of full houses across dealings, X: # full houses $\sim \text{Poisson}(1000 \times 0.0014)$

i.e. we are approximating binomial using Poisson

$\therefore P(X \geq 2) = 1 - P(X < 2) = 1 - P(X = 0) - P(X = 1) = 0.4082$