HOMEWORK 6 : Solutions
STAT 414 : Spring 2015

1) a) Assuming that the spins of the roulette wheel are independent,

$$P\left(\begin{array}{c}\text{Smith will lose}\\ \text{1st 5 bets}\end{array}\right) = P\left[(\text{lose 1st bet}) \text{ AND } (\text{lose 2nd bet}) \text{ AND } \cdots\cdots \text{ AND } (\text{lose 5th bet})\right]$$

$$= \left[P(\text{seeing numbers other than 1-12})\right]^5$$

$$= \left[\frac{26}{38}\right]^5 = 0.14995$$

b) Let X : Number of trials to the 1st win

Then X can be modeled as a geometric variable with $P(\text{success}) = P(\text{seeing a number between 1-12})$

$\therefore \quad X \sim \text{geometric}\left(\frac{12}{38}\right)$

$$\therefore P(X = x) = \left(\frac{12}{38}\right)^{x-1}\left(\frac{26}{38}\right) \qquad x = 1, 2, 3 \ldots$$

$$P\left(\begin{array}{c}\text{first win on}\\ \text{4th bet}\end{array}\right) = \left(\frac{12}{38}\right)^3\left(\frac{26}{38}\right) = 0.10115$$

2) X : Number of tails that occur before the 10th head

$\therefore$ we know that this can be modeled as a negative binomial variable.

Since the coin is fair, $P(\text{success}) = \frac{1}{2}$

$\therefore \quad X \sim NBD(10, \tfrac{1}{2})$

$$\therefore P(X = x) = {}^{9+x}C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{10} \qquad x = 0, 1, 2 \ldots$$

3) $P\left(\begin{array}{c}\text{we make exactly} \\ \text{'n' selections}\end{array}\right) = P\left[\left(\begin{array}{c}\text{First } n-1 \text{ selections} \\ \text{don't have } 2W-2B\end{array}\right) \text{ AND } \left(\begin{array}{c}\text{Last selection} \\ 2W-2B\end{array}\right)\right]$

Since we are replacing the balls after every selection, we can assume that the selections are independent

$$\therefore P\left(\begin{array}{c}\text{we make exactly} \\ \text{'n' selections}\end{array}\right) = \left(1 - \frac{{}^4C_2\,{}^4C_2}{{}^8C_4}\right)^{n-1} \frac{{}^4C_2\,{}^4C_2}{{}^8C_4}$$

$$= \left(\frac{17}{35}\right)^{n-1} \frac{18}{35}$$

4) Let X: # of defective items in a batch of randomly drawn 10 items

$$p(\text{success}) = P\left(\begin{array}{c}\text{finding defective} \\ \text{item}\end{array}\right) = \frac{6}{100} = 0.06$$

$$\therefore X \sim \text{Binomial}(10, 0.06)$$

$$p(X) = {}^{10}C_x\, 0.06^x\, 0.94^{10-x} \qquad x = 0, 1, 2 \dots$$

a) $P(X = 0) = {}^{10}C_0\, 0.06^0\, 0.94^{10}$

$$\boxed{P(X=0) = 0.5386}$$

b) $P(X > 2) = 1 - P(X \leq 2)$

$$= 1 - (P(X=0) + P(X=1) + P(X=2))$$

$$= 1 - {}^{10}C_0\, 0.06^0\, 0.94^{10} - {}^{10}C_1\, 0.06^1\, 0.94^9 - {}^{10}C_2 (0.06^2) 0.94^8$$

$$\boxed{P(X>2) = 0.1176}$$

☆ Many of you used a hypergeometric distribution, instead of binomial, for this problem. The key difference is in assuming whether the draws are made with or without replacement. If we assume that the overall population is large enough, we can correspondingly assume that drawing a sample of size 'n' is seemingly with replacement.

~~5) $f(x) = Cxe$~~

5) $f(x) = Cxe^{-x/2} I\{x>0\}$

Since the pdf must integrate to 1,

$$\int_0^\infty Ce^{-x/2}x\,dx = 1$$

Using integration by parts, we get,

$$C\left[\frac{xe^{-x/2}}{-1/2} - \frac{e^{-x/2}}{1/4}\right]_0^\infty = 1 \qquad e^{-\infty}=0 \ \& \ e^0=1$$

$$\therefore \frac{1}{C} = -0+0-0+4$$

$$\Rightarrow \boxed{C = \frac{1}{4}}$$

Now,

$$P(X \geq 5) = 1 - P(X<5)$$ In continuous distributions, $<$ & $\leq$ are the same

$$= 1 - C\int_0^5 xe^{-x/2}dx$$

$$= 1 - \frac{1}{4}\left[\frac{xe^{-x/2}}{-1/2} - \frac{e^{-x/2}}{1/4}\right]_0^5$$

$$= 1 - \frac{1}{4}\left[\frac{-5e^{-5/2}}{1/2} + 0 - \frac{e^{-5/2}}{1/4} + 4\right]$$

$$= -\frac{e^{-5/2}}{4}[-10-4]$$

$$\boxed{P(X\geq 5) = \frac{7e^{-5/2}}{2}} = 0.2873$$

6) $f(x) = \frac{10}{x^2} I\{x > 10\}$

a) $P(X > 20) = \int_{20}^{\infty} \frac{10}{x^2} dx = 1 - \int_{10}^{20} \frac{10}{x^2} dx$

$= 1 + \int_{10}^{20} \frac{-10}{x^2} dx$

$= 1 + 10 \left[\frac{1}{x}\right]_{10}^{20}$

$$\boxed{P(X > 20) = \frac{1}{2}}$$

b) $CDF(X) = P(X \le x)$

$= \int_{10}^{x} \frac{10}{x^2} dx$

$= \left[\frac{-10}{x}\right]_{10}^{x}$

$$\boxed{P(X \le x) = 1 - \frac{10}{x}} = F(x)$$

c) For a single device,

P(Device survives at least 15 hours) $= P(X \geq 15)$

$= 1 - P(X \le 15)$

$= 1 - F(15)$

$= 1 - 1 + \frac{10}{15}$

$= \frac{2}{3}$ //

Now, assuming that the survival of each device is independent of each other, we can model the survival of 6 devices as a binomial

$\therefore$ P(at least 3 will survive at least 15 hours) $= 1 - P($~~or~~ 0 will or 1 will or 2 will survive at least 15 hours$)$

$= 1 - {}^6C_0\left(\frac{2}{3}\right)^0\left(\frac{1}{3}\right)^6 - {}^6C_1\left(\frac{2}{3}\right)^1\left(\frac{1}{3}\right)^5 - {}^6C_2\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^4$

$= \boxed{0.8999}$

d) $E(X) = \int_{10}^{\infty} x \frac{10}{x^2} dx = 10 \int_{10}^{\infty} \frac{1}{x} dx = 10[\log x]_{10}^{\infty}$

$(\log \infty) = \infty$

$\therefore E(X) = \infty$ i.e. on an average, the device has an infinite lifetime

7) Say that the capacity must be 'k' thousand gallons

$0.01 = P(\text{supply exhausted in 1 week})$

$= P(x > k)$

$= \int_k^1 5(1-x)^4 dx$

$= -5\left[\frac{(1-x)^5}{5}\right]_k^1$

$= (1-k)^5 - 0$

$\Rightarrow k = 1 - \sqrt[5]{0.01}$

$\boxed{k = 0.6019}$

8) We know that,

$\int_0^1 x(a+bx^2)dx = \frac{3}{5}$ & $\int_0^1 (a+bx^2)dx = 1$

$\Rightarrow \left[\frac{ax^2}{2}\right]_0^1 + \left[\frac{bx^4}{4}\right]_0^1 = \frac{3}{5}$ & $[ax]_0^1 + \left[\frac{bx^3}{3}\right]_0^1 = 1$

$\Rightarrow \frac{a}{2} + \frac{b}{4} = \frac{3}{5}$ & $a + \frac{b}{3} = 1$

$\Rightarrow 2a + b = \frac{12}{5}$ & $3a + b = 3$

$\therefore$ $3a + b = 3$

$-\ 2a + b = \frac{12}{5}$

$\boxed{a = \frac{3}{5}}$

$\Rightarrow \boxed{b = \frac{6}{5}}$