HW 7 - Solutions

STAT 414 - Spring 2015

1) $f(x) = xe^{-x} I_{\{x>0\}}$ where X: Lifetime of electric tube (in hours)

$$E(X) = \int_0^\infty x^2 e^{-x}\,dx$$

$$= \left[-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\right]_0^\infty$$

$$\boxed{E(X) = 2}$$

Expected lifetime of a tube described in the problem is 2 hours.

2) <u>Interpretation</u>: Each point on the line segment may be chosen with equal probability

$\therefore$ For X: The location of a randomly chosen point on a line segment of length L

$\therefore X \sim \text{Uniform}(0, L) \Rightarrow f(x) = \frac{1}{L}$ for $0 \le X \le L$

<u>Required probability</u>:

Ratio of 1:4 $\rightarrow$ cut at $L/5$ or $4L/5$




For the ratio to be less than 1:4, the chosen point must be in one of the shaded regions

$$\therefore P(\text{ratio} < 1:4) = \int_0^{L/5} \frac{1}{L}\,dx + \int_{4L/5}^{L} \frac{1}{L}\,dx$$

$$= \frac{1}{L}\left[x\right]_0^{L/5} + \frac{1}{L}\left[x\right]_{4L/5}^{L}$$

$$= \boxed{\frac{2}{5}}$$

3) Let X: time of bus arrival during 0-30
and $X \sim \text{Uniform}(0, 30) \Rightarrow f(x) = \frac{1}{30} \quad 0 \le x \le 30$

a) $P(X > 10) = \int_{10}^{30} \frac{1}{30}\, dx = \frac{1}{30}[x]_{10}^{30} = \boxed{\frac{2}{3}}$

b) $P(X > 25 / X > 15) = \frac{P(X > 25 \cap X > 15)}{P(X > 15)} = \frac{\int_{25}^{30} \frac{1}{L} dx}{\int_{15}^{30} \frac{1}{L} dx}$

$= \frac{1/6}{1/2} = \boxed{\frac{1}{3}}$

4) Let X: Annual rainfall in the region (in inches)
$\therefore X \sim N(40, 4^2)$

We assume that the rainfall in any year is independent of rainfall during any other year

P( Rain fall more than 50 inches, in any given year ) $= P(X > 50)$
$= P\left(Z > \frac{50-40}{4}\right)$
$= P(Z > 2.5)$
$= 0.0062$ From the table

$\therefore$ P( 10 years before a year with $X > 50$) = P( 10 years with $X < 50$)
$= [1 - P(Z > 2.5)]^{10}$
$= \boxed{0.9396}$

5) $X \sim N(5, \sigma^2)$
$P(X > 9) = 0.2 \Longrightarrow P\left(Z > \frac{4}{\sigma}\right) = 0.2$

But $P(Z > 0.8416) = 0.2$

$\therefore \sigma = \frac{4}{0.8416} \Rightarrow \boxed{\sigma = 4.7527}$

6) P(Being in favour) = 0·65 , n = 100

let X: # in favour out of 100

$\therefore X \sim Binom(100, 0{\cdot}65)$

By normal approximation of binomial distribution,

$X \sim N(np, np(1-p))$ where $np = 100 \times 0{\cdot}65 = \underline{65 = \mu}$

$np(1-p) = 100 \times 0{\cdot}65 \times 0{\cdot}35 = \underline{22{\cdot}72 = \sigma^2}$

$\therefore X \sim N(65, 4{\cdot}7697^2)$

Since we are converting from discrete to continuous ~~as~~ distribution, we will have to add/subtract 0·5 appropriately

a) $P(X \geq 50) = P\left(z > \frac{49{\cdot}5 - 65}{4{\cdot}7697}\right)$

$= \boxed{0{\cdot}9994223}$

b) $P(60 \leq X \leq 70) = P\left(\frac{59{\cdot}5 - 65}{4{\cdot}7697} < z < \frac{70{\cdot}5 - 65}{4{\cdot}7697}\right)$

$= \boxed{0{\cdot}7511357}$

~~c) P(X ≤ 75) = P(z < 74·5~~

c) $P(X < 75) = P(X \leq 74)$

$= P\left(z < \frac{74.5 - 65}{4{\cdot}7697}\right)$

$= \boxed{0{\cdot}9768}$

7) Let $X_1$: # heads in 1000 tosses of a fair coin

$X_2$: # heads in 1000 tosses of a biased coin

$\therefore X_1 \sim Bin(100, 0{\cdot}5)$ , $X_2 \sim Binom(100, 0{\cdot}55)$

By normal approximation,

$X_1 \sim N(500, 15{\cdot}8114^2)$ and $X_2 \sim N(550, 15{\cdot}7321^2)$

$\therefore$ P(False conclusion for a fair coin) $=$ P(Greater than or equal to 525 for a fair coin)

$$= P(X_1 \geq 525)$$

$$= P\left(z > \frac{524.5 - 500}{15.8114}\right)$$

$$= \cancel{\boxed{0.1213}} \quad \boxed{0.060629}$$

P(False conclusion for a biased coin) $=$ P(Less than 525 heads for a biased coin)

$$= P(X_2 < 525)$$

$$= P\left(z < \frac{524.5 - 550}{15.7321}\right)$$

$$= \boxed{0.0525}$$

8) Let X: # heads in 10,000 independent tosses of a coin

$E(X) = 5000$

$$\cancel{P(X > 5799.5) = P\left(z > \frac{5799.5 - 5000}{50}\right) = P(Z > 15.99) \approx 0}$$

Normal approximation

$$P(X = 5800 \text{ for fair coin}) = P\left(\frac{5800.5 - 5000}{50} > Z > \frac{5799.5 - 5000}{50}\right)$$

$$\approx 0$$

Since the probability of this event occurring is almost zero for a fair coin, we can reasonably assume that the coin is not fair.

9) a) $E[|X-a|] = \int_a^A (x-a)\frac{dx}{A} + \int_0^a (a-x)\frac{dx}{A}$ — These are the intervals in which $|x-a|$ will be positive

$= \frac{A}{2} - \left[a - \frac{a^2}{A}\right]$

We want to minimize this

$\therefore \frac{d}{da}\left(E[|X-a|]\right) = \frac{2a}{A} - 1$. Equating to zero we get,

$\boxed{a = \frac{A}{2}}$

$\frac{d^2}{da}[E(|X-a|)] = 2 > 0 \Rightarrow$ minima is achieved

b) $E[|X-a|] = \int_0^a (a-x)\lambda e^{-\lambda x}dx + \int_a^\infty (x-a)\lambda e^{-\lambda x}dx$

$= a(1-e^{-\lambda a}) + ae^{-\lambda a} + \frac{e^{-\lambda a}}{\lambda} - \frac{1}{\lambda} + ae^{-\lambda a} + \frac{e^{-\lambda a}}{\lambda} - ae^{-\lambda a}$

Differentiating and equating to zero, we get

$e^{-\lambda a} = \frac{1}{2} \Rightarrow \boxed{a = \log\frac{2}{\lambda}}$

10) Let X: # years a radio functions $\sim \exp\left(\frac{1}{8}\right)$

~~P(X>8|X>~~

$P(X>11 \mid X>3) = P(X>8+3 \mid X>3) = P(X>8)$ By memorylessness

$= \int_8^\infty \frac{1}{8}e^{-x/8}dx = e^{-1} = \boxed{0.3679}$

11) $X \sim U(-1,1) \Rightarrow f(x) = \frac{1}{2}$

a) $P(|X|>1/2) = P(X>1/2 \text{ or } X<-1/2) = \int_{1/2}^{1}\frac{1}{2}dx + \int_{-1}^{-1/2}\frac{1}{2}dx = \boxed{\frac{1}{2}}$

b) $X \in (-1,1) \Rightarrow |X| \in (0,1)$

Since $X \sim U(-1,1)$, we can now imagine that $|X|$ has twice the density as before, at each point on $(0,1)$

$\therefore \boxed{f_{|X|}(a) = \frac{1}{1} = 1} \quad 0<a<1$

12) $X \sim \exp(1) \Rightarrow f(x) = e^{-x}$. $Y = \log X \Rightarrow x = e^y$

By theorem 7.1, considering $\left|\frac{d}{dy}e^y\right| = e^y$

$\therefore f_Y(y) = f_X(g^{-1}(y))\left|\frac{d}{dy}g^{-1}(y)\right|$

$\boxed{= e^{+y}\cdot e^{-e^y}}$