Homework 8: Solutions

STAT 414 : Spring 2015

1) a)

$X_2 \backslash X_1$	B 0	W 1
B 0	$\frac{8\times7}{12\times13} = \frac{14}{39}$	$\frac{8\times5}{13\times12} = \frac{10}{39}$
W 1	$\frac{8\times5}{13\times12} = \frac{10}{39}$	$\frac{5\times4}{13\times12} = \frac{5}{39}$

b) $$P(0,0,0) = \frac{8\times7\times6}{13\times12\times11} = \frac{28}{143}$$

$$P(0,0,1) = P(0,1,0) = P(1,0,0) = \frac{8\times7\times5}{13\times12\times11} = \frac{70}{429}$$

$$P(0,1,1) = P(1,0,1) = P(1,1,0) = \frac{8\times5\times4}{13\times12\times11} = \frac{40}{429}$$

$$P(1,1,1) = \frac{5\times4\times3}{13\times12\times11} = \frac{5\times4\times3}{13\times12\times11} = \frac{5}{143}$$

2) N_1 can possibly take values 1, 2, 3 or 4 and also N_2 can ~~e~~take values 1, 2, 3 or 4

$$P(N_1 = i, N_2 = j) = \frac{12}{120} \qquad i \,\&\, j \in (1,2,3,4) \text{ s.t. } i+j \leq 5$$

3) a) $$\int_0^1\int_0^2 \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) dy\,dx$$

$$= \frac{6}{7}\int_0^1 \left[x^2 y + \frac{xy^2}{4}\right]_0^2 dx$$

$$= \frac{6}{7}\int_0^1 \left[2x^2 + x\right] dx$$

$$= \frac{6}{7}\left[\frac{2x^3}{3} + \frac{x^2}{2}\right]_0^1$$

$$= \frac{6}{7}\left[\frac{2}{3} + \frac{1}{2}\right] = \boxed{1} \Longrightarrow$$ This is a valid joint distribution

②

b) $f_X(x) = \int_0^2 \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) dy$

$= \frac{6}{7}\left[x^2y + \frac{xy^2}{4}\right]_0^2$

$= \underline{f_X(x) = \frac{6}{7}[2x^2 + x]}$ $\quad 0 < x < 1$

★ specifying support of a variable, when writing a density, is very important

c) $P(X > Y) = \int_0^1 \int_0^x \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) dy\, dx$

$= \frac{6}{7} \int_0^1 \left[x^2y + \frac{xy^2}{4}\right]_0^x dx$

$= \frac{6}{7} \int_0^1 \left[x^3 + \frac{x^3}{4}\right] dx$

$= \frac{6}{7 \times 4}\left[\frac{5x^4}{4}\right]_0^1 = \boxed{\frac{15}{56} = P(X>Y)}$

d) $P(Y > 1/2 \mid X < 1/2) = \dfrac{P(Y > 1/2,\ X < 1/2)}{P(X < 1/2)}$

$= \dfrac{\int_{1/2}^2 \int_0^{1/2} \left(x^2 + \frac{xy}{2}\right) dx\, dy}{\int_0^{1/2} (2x^2 + x)\, dx}$

$= \dfrac{\int_{1/2}^2 \left[\frac{x^3}{3} + \frac{x^2y}{4}\right]_0^{1/2} dy}{\left[\frac{2x^3}{3} + \frac{x^2}{2}\right]_0^{1/2}}$

$= \dfrac{\int_{1/2}^2 \left[\frac{1}{24} + \frac{y}{16}\right] dy}{\left[\frac{2}{24} + \frac{1}{8}\right]}$

$= \dfrac{\left[\frac{y}{24} + \frac{y^2}{32}\right]_{1/2}^2}{5/24} = \dfrac{2/24 - 1/48 + 4/32 - 1/128}{5/24}$

$= \dfrac{23/128}{5/24} = \boxed{\frac{69}{80} = P(Y > 1/2 \mid X < 1/2)}$

e) $E(X) = \frac{6}{7}\int_0^1 (2x^3 + x^2)\,dx$

$= \frac{6}{7}\left[\frac{x^4}{2} + \frac{x^3}{3}\right]_0^1 = \frac{6}{7}\left[\frac{1}{2} + \frac{1}{3}\right] = \boxed{\frac{5}{7} = E(X)}$

f) $f_Y(y) = \frac{6}{7}\left[\frac{x^3}{3} + \frac{x^2 y}{4}\right]_0^1 = \left[\frac{1}{3} + \frac{y}{4}\right]\frac{6}{7} = \frac{4+3y}{14} \qquad 0 < y < 2$

$\therefore E(Y) = \int_0^2 \frac{6}{7}\left[\frac{y}{3} + \frac{y^2}{4}\right]dy$

$= \frac{6}{7}\left[\frac{y^2}{6} + \frac{y^3}{12}\right]_0^2 = \frac{6}{7}\left[\frac{4}{6} + \frac{8}{12}\right]$

$\boxed{E(Y) = \frac{8}{7}}$

4) Let X and Y denote the location of ambulance and accident respectively at the time of accident. We wand to evaluate,

$P\{|Y-X| < a\} = P(Y < X < Y+a) + P(X < Y < X+a)$

$= \frac{2}{L^2}\int_0^L \int_y^{\min(y+a,\,L)} dx\,dy$

The density is 1, since both the locations are random locations on the length

$= \frac{2}{L^2}\left[\int_0^{L-a}\int_y^{y+a} dx\,dy + \int_{L-a}^{L}\int_y^{L} dx\,dy\right]$

$= 1 - \frac{L-a}{L} + \frac{a}{L^2}\,t(L-a)$

$P\{|Y-X| < a\} = \frac{a}{L}\left(2 - \frac{a}{L}\right) \qquad 0 < a < L$

5) a) $\iint_{(x,y)\in R} f(x,y)\,dx\,dy = 1$

$$= \iint_{(x,y)\in R} c\,dy\,dx$$

$$= c \cdot A(R)$$

$$\Rightarrow \boxed{\text{Area} = \frac{1}{c}}$$

b) $f(x,y) = \frac{1}{4} \qquad -1 \le x,\ y \le 1$

$= f(x) \cdot f(y) \qquad$ where $f(v) = \frac{1}{2} \quad -1 \le v \le 1$

c) $P(X^2 + Y^2 \le 1) = \frac{1}{4} \iint_C dy\,dx$

$$= \frac{\text{area of circle}}{4} = \frac{\pi}{4}$$

6) Since all the points are chosen at random and equally likely to have been on the line anywhere, each of the points X_1, X_2, X_3 are equally likely to be the middle one

$\therefore$ $P(X_2$ lies between X_1 and $X_3) = \frac{1}{3}$ //

7) $f(x,y) = \frac{1}{x} \qquad 0 < y < x < 1$

$\int_0^1 \int_0^x \frac{1}{x}\,dy\,dx = \int_0^1 dx = 1 \Rightarrow$ this is a valid joint density

a) $\int_y^1 \frac{1}{x}\,dx = [\ln(x)]_y^1 = \boxed{-\ln(y) = f_Y(y)} \qquad 0 < y < 1$

b) $f_X(x) = \int_0^x \frac{1}{x}\,dy = \left[\frac{y}{x}\right]_0^x = \boxed{1 = f_X(x)} \qquad 0 < x < 1$

c) $E(X) = \int_0^1 x\,dx = \left[\frac{x^2}{2}\right]_0^1 = \boxed{\frac{1}{2} = E(X)}$

d) $E(Y) = -\int_0^1 y \cdot \ln(y)\, dy = \boxed{\frac{1}{4} = E(Y)}$

8) a) $f(x,y) = x + y \qquad 0 < x < 1, \; 0 < y < 1$

$f_X(x) = \left[xy + \frac{y^2}{2}\right]_0^1 = x + \frac{1}{2} \qquad 0 < x < 1$

$f_Y(y) = \left[\frac{x^2}{2} + xy\right]_0^1 = y + \frac{1}{2} \qquad 0 < y < 1$

$f_X(x) f_Y(y) = \left(x + \frac{1}{2}\right)\left(y + \frac{1}{2}\right) \neq x + y = f(x,y)$

$\therefore$ X and Y are not independent

b) $f_X(x) = x + \frac{1}{2} \qquad 0 < x < 1$

c) $P(X+Y<1) = \int_0^1 \int_0^{1-x} (x+y)\, dy\, dx$

$= \int_0^1 \left[xy + \frac{y^2}{2}\right]_0^{1-x} dx = \int_0^1 \left[x - x^2 + \frac{(1-2x+x^2)}{2}\right] dx$

$= \left[\frac{x^2}{2} - \frac{x^3}{3} + \frac{x}{2} - \frac{x^2}{2} + \frac{x^3}{6}\right]_0^1$

$= \frac{1}{2} + \frac{1}{2} - \frac{1}{3} - \frac{1}{2} + \frac{1}{6} = \boxed{\frac{1}{3} = P(X+Y<1)}$

9) $X \sim \exp(\lambda_1)$, $Y \sim \exp(\lambda_2)$. We will find distribution function.

$P(Z<a) = P\left(\frac{X_1}{X_2} < a\right) = \int_0^\infty \int_0^{ay} \lambda_1 e^{-\lambda_1 x} \lambda_2 e^{-\lambda_2 y}\, dx\, dy$

$= \int_0^\infty \lambda_1 \lambda_2 e^{-\lambda_2 y} \left[\frac{e^{-\lambda_1 x}}{-\lambda_1}\right]_0^{ay} dy = \frac{\lambda_1 \lambda_2}{-\lambda_1} \int_0^\infty e^{-\lambda_2 y}\left(e^{-a\lambda_1 y} - 1\right) dy$

$= -\lambda_2 \int_0^\infty \left[e^{-y(\lambda_2 + a\lambda_1)} - e^{-\lambda_2 y}\right] dy$

$= -\lambda_2 \left[\frac{e^{-y(\lambda_2 + a\lambda_1)}}{-(\lambda_2 + a\lambda_1)} - \frac{e^{-\lambda_2 y}}{-\lambda_2}\right]_0^\infty$

$= -\lambda_2 \left[\frac{1}{\lambda_2 + a\lambda_1} - \frac{1}{\lambda_2}\right] = \boxed{\frac{\lambda_1 a}{a\lambda_1 + \lambda_2} = F_Z(a)}$

b) $P(X_1 < X_2) = P\left(\frac{X_1}{X_2} < 1\right)$

$= \boxed{\frac{\lambda_1}{\lambda_1 + \lambda_2}}$ By considering $a=1$ in the previous part