Homework 9: Solutions
STAT 414: Spring 2015

1) a) $P(X=j, Y=i) = \frac{1}{5j}$ $\quad j=1,2 \cdots 5, \; i=1,2 \cdots j$

b) $P(X=j / Y=i) = \frac{P(X=j, Y=i)}{P(Y=i)}$

Now, $P(Y=i)$ needs to be considered for all scenarios in which $Y=i$ can occur i.e. whenever $i \leq j \leq 5$

$\therefore P(Y=i) = \sum_{k=i}^{5} \frac{1}{5k}$

$\therefore P(X=j / Y=i) = \frac{1/5j}{\sum_{k=i}^{5} 1/5k} = \frac{1}{j \sum_{k=i}^{5} \frac{1}{k}}$

c) As we can see, $P(X=j) = \frac{1}{5}$ for $j=1,2 \cdots 5$

$\Longrightarrow P(X=j / Y=i) \neq P(X=j)$

$\therefore$ X and Y are <u>not independent</u>

2) X: Smallest number observed on the dice

$\Longrightarrow X = 1, 2 \cdots 6$

Y: Largest number observed on the dice

$\Longrightarrow Y = 1, 2 \cdots 6$

Consider $X = i$, for $i = 1, 2 \cdots 6$,

Y can take values $i, i+1 \cdots 6$

Since both dice are fair, each outcome is equally likely

$\therefore P(Y=j \mid X=i) = \frac{1}{(6-i)+1}$ $\quad$ For $i=1,2 \cdots 6$, $i \leq j \leq 6$

Since the possible values of Y depend on values of X, $X \not\perp\!\!\!\perp Y$. Also. $P(Y=j) = \frac{1}{6} \neq \frac{1}{(6-i)+1} = P(Y=j / X=i)$

3) We have the following joint density

		$X = 1$	$X = 2$	
Y	1	$1/8$	$1/8$	$1/4$
	2	$1/4$	$1/2$	$3/4$
		$3/8$	$5/8$	

We know that:

$$P(X=x/Y=i) = \frac{P(X=x, Y=i)}{P(Y=i)}$$

a) We want to find a function for

$$E(X/Y=i) = \sum_{x=1}^{2} x P(X=x/Y=i) \qquad i=1,2$$

For $\underline{Y=1}$,

$$E(X/Y=1) = \sum_{x=1}^{2} x P(X=x/Y=1)$$

$$= 1 * P(X=1/Y=1) + 2 * P(X=2/Y=1)$$

$$= \frac{P(X=1, Y=1)}{P(Y=1)} + \frac{2P(X=2, Y=1)}{P(Y=1)}$$

$$= \frac{1}{8} / \frac{1}{4} + \frac{2}{8} / \frac{1}{4} = \frac{3}{8} \div \frac{1}{4}$$

$$\boxed{E(X/Y=1) = \frac{3}{2}}$$

For $\underline{Y=2}$,

$$E(X/Y=2) = \sum_{x=1}^{2} x P(X=x/Y=2)$$

$$= \left(\frac{1}{4} \div \frac{3}{4}\right) + \left(\frac{2}{2} \div \frac{3}{4}\right)$$

$$= \frac{1}{3} + \frac{4}{3}$$

$$\boxed{E(X/Y=2) = \frac{5}{3}}$$

b) We can verify that X and Y are not independent with a simple example

$P(X=1, Y=1) = \frac{1}{8}$

$P(X=1) \cdot P(Y=1) = \frac{1}{4} \times \frac{3}{8} = \frac{3}{32} \neq \frac{1}{8}$

Since for one instance $P(X=i, Y=j) \neq P(X=i)\,P(Y=j)$, X and Y are not independent

c) i) $P(XY \leq 3)$: $XY \leq 3$ includes all cases except $X=2$ & $Y=2$, since $2 \times 2 = 4$

$\therefore P(XY \leq 3) = P(X=1, Y=1) + P(X=1, Y=2) + P(X=2, Y=1)$

$= \frac{1}{8} + \frac{1}{8} + \frac{1}{4}$

$$\boxed{P(XY \leq 3) = \frac{1}{2}}$$

ii) $P(X+Y > 2) = 1 - P(X+Y=2)$ Since $X+Y \geq 2$ always

$= 1 - \frac{1}{8}$

$$\boxed{P(X+Y > 2) = \frac{7}{8}}$$

iii) $P\left(\frac{X}{Y} > 1\right) = P(X=2, Y=1)$

$$\boxed{P\left(\frac{X}{Y} > 1\right) = \frac{1}{8}}$$

4) $f(x,y) = x e^{-x(y+1)}$ $x > 0, y > 0$

a) $\underline{f_{X|Y}(x|y)} = \frac{f(x,y)}{f_Y(y)} = \frac{x e^{-x(y+1)}}{\int_0^\infty x e^{-x(y+1)}\,dx}$

$$= \frac{x e^{-x(y+1)}}{\left[\frac{x e^{-x(y+1)}}{-(y+1)} - \frac{e^{-x(y+1)}}{[-(y+1)]^2}\right]_0^\infty} = \boxed{(y+1)^2 x e^{-x(y+1)}}$$

For $x > 0$

$$f_{Y|X}(y|x) = \frac{x e^{-x(y+1)}}{\int_0^{\infty} x e^{-x(y+1)}\,dy} = \frac{x e^{-x(y+1)}}{\left[\frac{x e^{-x(y+1)}}{-x}\right]_0^{\infty}}$$

$$= \frac{x e^{-x(y+1)}}{e^{-x}}$$

$$\boxed{f_{Y|X}(y|x) = x e^{-xy}} \qquad y > 0$$

b) $F(z) = F(XY) = P(XY < a) = 1 - e^{-a}$

$\therefore f_Z(z) = f_{XY}(a) = \underline{\underline{e^{-a}}}$ $\qquad a > 0$

5) $f(x,y) = c(x^2 - y^2)e^{-x} \qquad 0 < x < \infty, \; -x < y < x$

$$f_{Y|X}(y|x) = \frac{f(y,x)}{f_X(x)}$$

Since this ratio will cancel out the c, we need not separately evaluate c

$$f_{Y|X}(y|x) = \frac{(x^2 - y^2)e^{-x}}{\int_{-x}^{x} (x^2 - y^2)e^{-x}\,dy} = \frac{(x^2 - y^2)e^{-x}}{\left[x^2 e^{-x} y - \frac{e^{-x} y^3}{3}\right]_{-x}^{x}}$$

$$= \frac{(x^2 - y^2)e^{-x}}{\left[\frac{6x^3 e^{-x} - 2x^3 e^{-x}}{3}\right]} = \boxed{\frac{3}{4x^3}(x^2 - y^2)} \qquad -x < y < x$$

$$\therefore F_{Y|X}(y|x) = \frac{3}{4x^3} \int_{-x}^{y} (x^2 - y^2)\,dy$$

$$= \frac{3}{4x^3}\left[x^2 y - \frac{y^3}{3} + \frac{2x^3}{3}\right] \qquad -x < y < x$$

6) We want to find,

$$f(\lambda | n) = \frac{P(N=n | \lambda)\, g(\lambda)}{P(N=n)} \qquad \text{Bayes' formula}$$

$$= c_1 e^{-\lambda} \lambda^{n} \alpha e^{-\alpha\lambda} (\alpha\lambda)^{s-1}$$

$$= c_2 e^{-(\alpha+1)\lambda} \lambda^{n+s-1}$$

Here, c_1 and c_2 are constants such that they don't depend on λ.

Ignoring the constant, we can identify that $f(\lambda | n)$ is a gamma density with parameters $(\alpha+1, n+s)$

$\therefore$ $E(\#\text{ accidents / insured}) = E(f(\lambda | n))$

$= E(\text{gamma density})$

$= \boxed{\dfrac{n+s}{\alpha+1}}$