Math/Stat 414.4 Spring 2015 Exam 2 Name: ________

Instructions: Please read!

1. Do all work on this exam packet. It is okay to leave your answer unsimplified, as in $0.56\frac{15!}{6!9!}$ or $14e - 20$. This means that no calculator is needed. If you would like to express a probability concerning the standard normal distribution, use $\Phi(x)$ as the symbol for the cdf function of a standard normal evaluated at x.
2. Show all work for full credit! Small mistakes in arithmetic will not reduce credit if you show work; conversely, even a correct answer could get no credit without supporting work.
3. I will award partial credit where appropriate. Each problem/subproblem is worth 11 points.

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1. Suppose that the average number of cars abandoned weekly on a certain highway is 2.2. Approximate the probability that there will be no abandoned cars in the next week. State your assumptions.

The number of cars on the highway is assumed large (n is large). Each car breaks down with a small probability, and is independent of every other car (p is small). So, the number of cars to break down is binomial with a large n and small p and is thus approximated by a Poisson R.V. (with $\lambda = 2.2$)

$$P(X=0) \approx \frac{e^{-\lambda}\lambda^0}{0!} = e^{-2.2}$$

2. A ball is drawn from an urn containing 3 white and 4 black balls. After the ball is drawn, it is replaced and another ball is drawn. This process goes on indefinitely.

   (a) Let X be the number of white balls in the first four draws. What is the name we have given to this type of random variable, and what is the probability that $X = 3$?

X is binomial with $n=4$ and $p=3/7$ (proportion of white balls)

$$P(X=3)=\binom{4}{3}\left(\frac{3}{7}\right)^3\frac{4}{7}$$

$$=4\left(\frac{3}{7}\right)^3\frac{4}{7}$$

   (b) Let Y be the number of draws required to obtain the first black ball. What is the name we have given to this type of random variable, and what is EY?

Y is geometric with $p=\frac{4}{7}$

For a geometric $EY=\frac{1}{p}$. So, $EY=\frac{7}{4}$

   (c) Suppose that you draw 150 times. What is the approximate probability that more than 90 of the draws are white balls?

$n=150$ $p=3/7$ X is binomial

$$P(X>90)=P\left(\frac{X-\frac{450}{7}}{\sqrt{\frac{150(4)(3)}{7^2}}}>\frac{90-\frac{450}{7}}{\frac{\sqrt{150(12)}}{7}}\right)$$

$$=1-\Phi\left(\frac{630-450}{\sqrt{150(12)}}\right)$$

3. Suppose that X is a continuous random variable which has the following density

$$f(x) = \frac{2}{9} x I_{\{0<x<3\}}$$

(a) Find $P(1 < X < 2)$.

$$P(1<X<2) = \int_1^2 \frac{2}{9} x I_{\{0<x<3\}}\, dx$$
$$= \int_1^2 \frac{2}{9} x \, dx$$
$$= \frac{1}{9} x^2 \Big|_1^2 = \frac{4}{9} - \frac{1}{9} = \frac{1}{3}$$

(b) Find $E[X]$.

$$EX = \int_0^3 x \frac{2}{9} x \, dx$$
$$= \frac{2}{9} \frac{1}{3} x^3 \Big|_0^3$$
$$= \frac{2}{27}(27) = 2$$

(c) Find $Var[X]$. $\quad Var\, X = EX^2 - (EX)^2$

$$EX^2 = \int_0^3 \frac{2}{9} x^3 \, dx$$
$$= \frac{1}{18} x^4 \Big|_0^3$$
$$= \frac{81}{18} = \frac{9}{2}$$

$$= \frac{9}{2} - (2)^2$$
$$= \frac{1}{2}$$

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$$= \int_1^2 \frac{2}{9}x\,dx$$

$$= \frac{1}{9}x^2\Big|_1^2 = \frac{4}{9} - \frac{1}{9} = \frac{1}{3}.$$

(b) Find $E[X]$.

$$EX = \int_0^3 x\frac{2}{9}x\,dx$$

$$= \frac{2}{9}\frac{1}{3}x^3\Big|_0^3$$

$$= \frac{2}{27}(27) = 2$$

(c) Find $Var[X]$. $\quad Var X = EX^2 - (EX)^2$

$$EX^2 = \int_0^3 \frac{2}{9}x^3\,dx$$

$$= \frac{1}{18}x^4\Big|_0^3$$

$$= \frac{81}{18} = \frac{9}{2}$$

$$= \frac{9}{2} - (2)^2$$

$$= \frac{1}{2}$$

(d) Find the moment generating function of X. (Hint: Integration by parts!)

$u = x \quad du = dx$

$v = \frac{1}{t}e^{xt} \quad dv = e^{xt}dx$

$$M(t) = Ee^{Xt} = \int_0^3 e^{xt} \frac{2}{9} x \, dx$$

$$= \frac{2}{9} \int_0^3 e^{xt} x \, dx$$

$$= \frac{2}{9}\left[\frac{x}{t} e^{xt} \Big|_0^3 - \int_0^3 \frac{e^{xt}}{t} dx \right]$$

$$= \frac{2}{9}\left[\frac{3e^{3t}}{t} - \frac{e^{3t}-1}{t^2} \right]$$

$$= \frac{2}{9}\left[\frac{1}{t^2} - \frac{e^{3t}}{t^2} + \frac{3e^{3t}}{t} \right]$$

(e) Find the density of $Y = X^2$.

$0 < y < 9$

$$F_Y(y) = P(Y \le y)$$

$$= P(X^2 \le y)$$

$$= P(X \le \sqrt{y})$$

$$= \int_0^{\sqrt{y}} \frac{2}{9} x \, dx$$

$$= \frac{1}{9} x^2 \Big|_0^{\sqrt{y}}$$

$$= \frac{1}{9} y$$

density

$$\frac{d}{dy} F_Y(y) = \frac{d}{dy}\left(\frac{1}{9} y\right)$$

$$= \frac{1}{9}$$