Stat 440 Spring 2015 Midterm Exam Name: ________

Instructions: Please read!

1. Do all work on this exam packet. It is okay to leave your answer unsimplified, as in $0.56\frac{15!}{6!9!}$ or $14e-20$. This means that no calculator is needed.
2. Show all work for full credit! Small mistakes in arithmetic will not reduce credit if you show work; conversely, even a correct answer could get no credit without supporting work.
3. I will award partial credit where appropriate.

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1. Write an algorithm to return a Monte Carlo estimate of the integral

$$\int_2^5 x^{3/2}e^{-x}\,dx$$

The estimate should be given in terms of a 95% confidence interval for the integral.

Let $g(x) = x^{3/2} e^{-x}$

- generate m uniform RV $U_i \sim unif(0,1)$
- convert to uniform on $(2,5)$

  $V_i = 3U_i + 2$
- Find $g(V_i)$
- $\hat{\theta} = \frac{1}{m}\sum_{i=1}^{m} 3g(V_i)$
- Find sample variance of the $3g(V_i)$

  $s^2 = \frac{1}{m}\sum_{i=1}^{m}(3g(V_i) - \hat{\theta})^2$
- $\hat{\theta} \pm 2\sqrt{\frac{s^2}{m}}$ to return

2. Suppose that you would like to calculate the expected value for a Weibull distribution with parameters $a = 2$ and $\beta = 3$. The cdf for a Weibull is

$$F(t) = 1 - e^{-at^\beta} \quad t > 0$$

(a) Write down the integral that needs to be calculated. (You do not need to solve; just write it down.)

$$f(t) = F'(t) = a\beta t^{\beta-1} e^{-at^\beta} \quad t > 0$$

$$ET = \int_0^\infty a\beta t^\beta e^{-at^\beta} \, dt$$

$$= \int_0^\infty 6t^3 e^{-2t^3} \, dt$$

(b) Suppose that you can simulate only standard exponential random variables. Give an algorithm for estimating the integral in the previous part using these simulated exponential random variables.

Use importance Sampling

$$\int_0^\infty 6t^3 e^{-2t^3} \, dt = \int_0^\infty \frac{6t^3 e^{-2t^3}}{e^{-t}} \, e^{-t} \, dt$$

- Generate std. exponential RV's $V_1 - V_m$
- Calculate estimate

$$\frac{1}{m} \sum_{i=1}^{m} \frac{6V_i e^{-2V_i^3}}{e^{-V_i}}$$

3. We need to simulate from a random variable with the following DENSITY:

$$f(x) = \begin{cases} \frac{1}{2}(x-2) & 2 \le x \le 3 \\ \frac{1}{2}\left(2-\frac{x}{3}\right) & 3 < x \le 6 \\ 0 & otherwise \end{cases}$$

Give an algorithm using each of the following methods:

(a) inverse CDF method

$2 \le x \le 3$ $\quad F(x) = \int_2^x \frac{1}{2}(u-2)\,du$

$$= \frac{1}{2}\left[\frac{1}{2}u^2 - 2u\right]_2^x$$

$$= \frac{1}{4}x^2 - x + 1$$

$3 < x < 6$ $\quad F(x) = \int_3^x \frac{1}{2}\left(2-\frac{u}{3}\right)du + \frac{1}{4}$ F(3)

$$= \frac{1}{2}\left(2u - \frac{u^2}{6}\right)\Big]_3^x + \frac{1}{4}$$

$$= x - \frac{x^2}{12} - 2$$

Simulate $U \sim Unif(0,1)$

If $U < \frac{1}{4}$, Solve $U = \frac{1}{4}x^2 - x + 1$ for $x \in (2,3)$

$U > \frac{1}{4}$, solve $U = x - \frac{x^2}{12} - 2$ for $x \in (3,6)$

(b) accept/reject method

Take $g(x)$ to be unif from $(2,6)$

$$g(x) = \frac{1}{4}$$

So,

$$\max \frac{f(x)}{g(x)} = \frac{\frac{1}{2}}{\frac{1}{4}} = 2 \leftarrow c$$

Generate U. Convert $V = 4U + 2$

Generate $U' \sim Unif(0,1)$.

If $U' \leq \frac{f(V)}{g(V)\,2}$, accept V. Otherwise, return to top.