Supplemental Problems for Continuous Time Markov Chains

1. Let X_n be a discrete time Markov chain with probability transition matrix
$$P = \begin{pmatrix} 0.5 & 0.3 & 0.2 \\ 0.2 & 0.6 & 0.2 \\ 0.2 & 0.4 & 0.4 \end{pmatrix}$$
and $N(t)$ is a Poisson process with $\lambda = 3$. $X_{N(t)}$ is now a continuous time Markov chain.
   (a) Find the $P^*_{i,j}$ and v_i for $X_{N(t)}$.
   (b) Find the transition rate matrix Q for $X_{N(t)}$

2. A population of organisms consists of both male and female members. In a small colony any particular male is likely to mate with any particular female in any time interval of length h, with probability $\lambda h + o(h)$. Each mating immediately produces one offspring, equally likely to be male or female. Let $X_1(t)$ and $X_2(t)$ denote the number of males and females in the population at t. Derive the transition rates of the continuous time Markov chain $(X_1(t), X_2(t))$ and also derive the appropriate P^* and v_i values.

3. There are N individuals in a population, some of whom have a certain infection that spreads as follows. Contacts between two members of this population occur in accordance with a Poisson process having rate λ. When a contact occurs, it is equally likely to involve any of the $\binom{N}{2}$ pairs of individuals in the population. If a contact involves an infected and a noninfected individual, then with probability p the noninfected individual becomes infected. Once infected, an individual remains infected from that time on. Let $X(t)$ denote the number of infected members of the population at time t. Specify the transition rates and the P^* and v_i values.

4. A job shop consists of three machines and two repairman. The amount of time a machine works before breaking down is exponentially distributed with mean 10. If the amount of time it takes a single repairman to fix a machine is exponentially distributed with mean 8, then

   (a) Write down the rate matrix for this example

   (b) What are the v_i and the $P^*_{i,j}$ for this example?

5. Two people are working in a small office selling shares in a mutual fund. Each is either on the phone or not. Suppose salesman one is on the phone for an exponentially distributed amount of time with rate $\mu_1 = 2$. Salesman two is on the phone for an exponentially distributed amount of time with rate $\mu_2 = 3$. The rate for an individual salesman to stay off the phone is $\lambda = 3$.

   (a) Write down the Q matrix for a continuous time Markov chain representing WHICH salesman is on the phone. (The state space coould be described with $\{0, 1, 2, 12\}$, for example.)

   (b) Can one easily formulate a Markov chain, $X(t)$, which represents only the number of salesmen on the phone? If so, write down the new Q matrix for this modified chain. If not, then why not?

   (c) Find the v_i and P^*_{ij} for the continuous time Markov chain in (a).

6. Two frogs are playing near a pond. When they are in the sun they get too hot and individually jump in the lake at rate 1 frog per hour. When they are in the lake they get too cold and each jump onto the land at rate 2. Let $X(t)$ be the number of frogs in the sun.

   (a) Find the Q matrix for $X(t)$.

   (b) If there are two frogs in the sun, what is the precise distribution of the time until one of the frogs jumps into the water? (i.e. Write the probability density for this time.)

   (c) Write down the model in terms of rates to leave a particular state (i.e. v_0, v_1, v_2) and the probabilities of moving to a particular state given that a jump has occurred (i.e. the P^*_{ij}).

7. Suppose that you have a continuous time Markov chain with transition matrix

$$Q = \begin{pmatrix} -3 & 1 & 2 \\ 2 & -4 & 2 \\ 4 & 3 & -7 \end{pmatrix}$$

(a) What are the rates for leaving each state (i.e. the v_i's) and the probabilities of moving to other states (given that you are leaving state i), in other words, the $P^*_{i,j}$.

(b) If $N(t)$ is a Poisson distribution with $\lambda = 8$, find the probability transition matrix of a discrete time Markov chain X_n so that $X_{N(t)}$ has transition matrix Q.