Math/Stat 416.2 Quiz 1 Name: ____________________

Instructions: Please read!

1. Do all work on this exam packet. It is okay to leave your answer unsimplified, as in $0.56\frac{15!}{6!9!}$ or $14e - 20$. This means that no calculator is needed.
2. Show all work for full credit! Small mistakes in arithmetic will not reduce credit if you show work; conversely, even a correct answer could get no credit without supporting work.
3. I will award partial credit where appropriate.
4. You may use all the facts from the probability review as given facts UNLESS you are explicitly asked to derive something.

---

1. (25 pts) The lifetime of a special type of battery is a random variable with mean 40 hours and standard deviation 20 hours. A battery is used until it fails, at which point it is replaced by a new one. Assuming a stockpile of 25 such batteries, the lifetimes of which are independent, approximate the probability that over 1100 hours of use can be obtained. (Express you answer using $\Phi(\cdot)$, the CDF of a standard normal random variable.)

   Solution: We first express the probability as a sum of the T_i, the lifetime of battery, i

   $$P\left(T_1 + ... + T_{25} > 1100\right)$$

   We note that the probability of sums of iid random variables may be approximated by a normal random variable (via the CLT). So, we standardize

   $$P\left(\frac{T_1 + ... + T_{25} - 25(40)}{20\sqrt{25}} > \frac{1100 - 25(40)}{20\sqrt{25}}\right)$$

   This probability is approximately (by CLT and a little bit of arithmetic)

   $$1 - \Phi(1)$$

2. (25 pts) Suppose that the number of accidents occurring on a highway each day is a Poisson random variable with parameter $\lambda = 3$.

   (a) Find the probability that 3 or more accidents occur today. Express your answer in closed form, i.e. not in terms of a series/infinite sum.

   solution: The number of accidents we denote by N. So,

$$P(N \geq 3) = 1 - P(N \leq 2) = 1 - e^{-3} - 3e^{-3} - 9/2e^{-3}$$

   (b) Repeat part (a) under the condition that at least 1 accident occurs today.

   solution: Now, we need to condition on the event that $N > 1$. So,

$$P(N \geq 3 | N > 1) = \frac{P(N \geq 3 \ \& \ N \geq 1)}{P(N \geq 1)} = \frac{P(N \geq 3)}{P(N \geq 1)} = \frac{1 - e^{-3} - 3e^{-3} - 9/2e^{-3}}{1 - e^{-3}}$$

3. (25 pts) Suppose that random variable$X$ has a uniform density over the interval $[-1, 1]$.

(a) Derive the variance for X. (Show the derivation!)

solution: The mean of X is zero.

$$\int_{-1}^{1} 1/2x dx = \frac{1}{4}x^2 \Big|_{-1}^{1} = 0$$

So, we need to find EX^2, which will then be equal to the variance.

$$\int_{-1}^{1} 1/2x^2 dx = \frac{1}{6}x^3 \Big|_{-1}^{1} = \frac{1}{3}$$

(b) Derive the moment generating function for X. (Show the derivation!)

solution:

$$M_X(t) = E[e^{Xt}] = \int_{-1}^{1} \frac{1}{2} e^{xt} dx = \frac{1}{2t} e^{xt} \Big|_{-1}^{1} = \frac{e^t - e^{-t}}{2t}$$

(Note that technically this does not exist at t=0; however, if you define the mgf to be one at zero, then things are fine.)

4. (25 pts) Suppose that the number of cars arriving to a (very) rural intersection per hour is Poisson distributed with $\lambda = 2$, and the number of cars per hour is independent of the number of cars in any other hour.

   (a) Suppose you observe this intersection. What is the probability that the first time you see EXACTLY two cars in one hour happens in the fourth hour of watching the intersection?

   solution: The number of hours X to the first "success" corresponds to a geometric random variable with $p = P(N = 2) = e^{-2}2^2/2$ where N is Poisson with $\lambda = 2$. So,

$$P(X = 4) = (1 - p)^3 p = (1 - 2e^{-2})^3 2e^{-2}$$

   (b) On average, how many hours do you need to watch the intersection before getting an hour with EXACTLY two cars arriving during that hour?

   solution: As noted in part a, X has a geometric distribution so, $EX = 1/p = e^2/2$.