Math/Stat 416.2 Quiz 3 Name: ____________________

Instructions: Please read!

1. Do all work on this exam packet. It is okay to leave your answer unsimplified, as in $0.56\frac{15!}{6!9!}$ or $14e - 20$. This means that no calculator is needed.
2. Show all work for full credit! Small mistakes in arithmetic will not reduce credit if you show work; conversely, even a correct answer could get no credit without supporting work.
3. I will award partial credit where appropriate.
4. You may use all the facts from the probability review as given facts UNLESS you are explicitly asked to derive something.

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1. (25pts) Identify the communication classes for the Markov chain with the following probability transition matrix, THEN say whether each is transient or recurrent. (So, there is not confusion, let's denote the states by $\{1, 2, 3, 4, 5, 6\}$.)

$$\begin{pmatrix} 2/3 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 1/3 & 1/3 & 1/3 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 & 0 \end{pmatrix}$$

solution: The communication classes are $\{1, 4\}$, $\{2, 5\}$, $\{3\}$, and $\{6\}$. $\{1, 4\}$ is recurrent. $\{2, 5\}$ is recurrent. $\{3\}$ and $\{6\}$ are each transient.

2. (25 pts) Suppose that Y has an exponential distribution with the following density

$$f_Y(y) = \frac{1}{2}e^{-y/2}$$

for $y > 0$. The conditional distribution of random variable X given $Y = y$ has a Poisson distribution with parameter $\lambda = y$.

(a) Find the $P(X = 1)$.
solution: $P(X = 1) = E[P(X = 1|Y)] = E[Ye^{-Y}]$ So,

$$E[Ye^{-Y}] = \int_0^\infty ye^{-y}1/2e^{-y/2}dy = \frac{1}{2}\frac{2}{3}\int_0^\infty ye^{-3y/2}3/2dy = \frac{1}{3}\frac{2}{3} = \frac{2}{9}$$

using the fact that the mean of an exponential distribution is one over the rate.

(b) Find the variance of X.
solution: Recall: $Var[X] = E[Var[X|Y]] + Var[E[X|Y]]$ So,

$$Var[X] = E[Y] + Var[Y] = 1/2 + 1/4 = 3/4$$

3. (25 pts) Recall the example from the last quiz: A taxicab driver moves between the airport A and two hotels, B and C, according to the following rules. If he is at the airport, then he will go to one of the two hotels next with equal probability. If he is at one of the hotels, then he will go to the airport with probability 3/4 and to the other hotel with probability 1/4. The probability transition matrix is then

$$P = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 3/4 & 0 & 1/4 \\ 3/4 & 1/4 & 0 \end{pmatrix}$$

Find the limiting distribution for this Markov chain.

solution: First, we write down the equations $\pi = \pi P$

$$\begin{aligned} \pi_1 &= 3/4\pi_2 + 3/4\pi_3 \\ \pi_2 &= 1/2\pi_1 + 1/4\pi_3 \\ \pi_3 &= 1/2\pi_1 + 1/4\pi_2 \end{aligned}$$

(This is but one way to now solve these.) Eliminate the first equation for $1 = \pi_1 + \pi_2 + \pi_3$. Note the symmetry in the last two equations. Substitute the third into the second equation to obtain

$$\frac{15}{16}\pi_2 = \frac{5}{8}\pi_1$$

which implies $\pi_2 = 2/3\pi_1$. By the symmetry, $\pi_3 = 2/3\pi_1$. Now using the fact that they sum to one, we get $2/3\pi_1 + 2/3\pi_1 + \pi_1 = 1$. Thus, $\pi_1 = 3/7$ which means that $\pi_2 = \pi_3 = 2/7$.

4. (25 pts) A standard six-sided die is rolled until either a 1 is obtained (and you win!) or a 5 or 6 is obtained (where you lose). If another number comes up (i.e. a 2, 3 or 4), then you continue to roll. Answer the following–as always, be sure to give clear reasoning to support your answers.

   (a) Find the expected number of rolls in order to either win or lose.

   solution: Let T be the random variable for the number of rolls up to and including a win or loss. Condition on what happens on the first roll

   $$E[T] = E[T|A_1]P(A_1) + E[T|A_{5,6}]P(A_{5,6}) + E[T|A_{2,3,4}]P(A_{2,3,4})$$

   which can be written as

   $$E[T] = 1 \cdot 1/6 + 1 \cdot 2/6 + (E[T] + 1)3/6$$

   Solve for $E[T]$ to arrive at 2.

   (b) Find the probability that you eventually win.

   solution: Let B be the even that you win and condition on what happens at the first roll.

   $$P(B) = P[B|A_1]P(A_1) + P[B|A_{5,6}]P(A_{5,6}) + P[B|A_{2,3,4}]P(A_{2,3,4})$$

   We can convert this into

   $$P(B) = 1 \cdot 1/6 + 0 \cdot 2/6 + P(B)3/6$$

   So, solving for $P(B)$ we arrive at a 1/3 probability of winning.