Math/Stat 416.2 Quiz 4 Name: ____________________

Instructions: Please read!

1. Do all work on this exam packet. It is okay to leave your answer unsimplified, as in $0.56\frac{15!}{6!9!}$ or $14e - 20$. This means that no calculator is needed.
2. Show all work for full credit! Small mistakes in arithmetic will not reduce credit if you show work; conversely, even a correct answer could get no credit without supporting work.
3. I will award partial credit where appropriate.
4. You may use all the facts from the probability review as given facts UNLESS you are explicitly asked to derive something.

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1. One hundred lightbulbs are simultaneously put on a test to determine time to failure. Suppose that the lifetimes of the individual lightbulbs are independent exponential random variables with mean 200 hours.

   (a) Write down the probability density function for the time until the first lightbulb fails.

   solution: The minimum time of 100 lightbulbs each with an exponential distribution with $\lambda = 1/200$ is also exponential with rate 100/200. So, the density for the time to the first is

   $$f(x) = 1/2e^{-1/2x}$$

   for $x > 0$.

   (b) Find the expected time until the third lightbulb fails.

   solution: After the first one fails, there are only 99 bulbs. After the second one fails, there are 98. So,

   $$E(T_1 + T_2 + T_3) = ET_1 + ET_2 + ET_3 = \frac{200}{100} + \frac{200}{99} + \frac{200}{98}$$

2. Answer the two following problems about these different branching processes.

   (a) Suppose that Y_n is a branching process with the distribution of offspring (what we called Z in class) described by

   $$P(Z=0)=3/8 \qquad P(Z=1)=1/2 \qquad P(Z=2)=1/8$$

   What is the mean of Z What does this imply about EX_n as $n \to \infty$? Find the eventual probability of extinction.

   solution: The mean of Z is $0 \cdot 3/8 + 1 \cdot 1/2 + 2 \cdot 1/8$, which is $3/4$. Since, $EX_n = \mu_Z^n$ and μ is less than one, then EX_n converges to zero. Since μ_z is less than one, we know from class that the probability of extinction is one.

   (b) Suppose that X_n is a branching process with the distribution of offspring (what we called Z in class) described by

   $$P(Z=0)=1/8 \qquad P(Z=1)=1/2 \qquad P(Z=2)=3/8$$

   What is the mean of Z? What does this imply about EX_n as $n \to \infty$? Find the eventual probability of extinction.

   solution: The mean of Z is $0 \cdot 1/8 + 1 \cdot 1/2 + 2 \cdot 3/8$, which is $5/4$. Since, $EX_n = \mu_Z^n$ and μ is greater than one, then EX_n increases without bound. Since μ_z is greater than one, we know that the extinction probability is the smallest solution to the equation $\pi = \phi_z(\pi)$. First

   $$\phi_z(t) = 1/8 + 1/2t + 3/8t^2$$

   So, we need to find the roots of

   $$\pi = 1/8 + 1/2\pi + 3/8\pi^2$$

   These are $1/3, 1$. So, the probability of extinction is $1/3$.

3. Suppose that a discrete random variable X has a probability generating function (also called a factorial moment generating function) of

$$\phi(t) = \frac{0.25}{1 - 0.75t}$$

(a) Find $P(X = 2)$

solution: Take derivative twice, evaluate at zero, then divide by two.

$$\phi''(t) = 2(0.25)(0.75)^2(1 - 0.75t)^{-3}$$

So, $\phi''(0) = 2(0.25)(0.75)^2$. $P(X = 2) = \phi''(0)/2 = (0.25)(0.75)^2$.

(b) Find EX

solution: To find the expected value, we take the derivative once and evaluate at one.

$$\phi'(t) = (0.25)(0.75)(1 - 0.75t)^{-2}$$

So, $EX = \phi'(1) = 0.75/0.25$.

4. Suppose that you arrive at a single-teller bank to find five other customers in the bank, one being served (so, remember you'll need to wait for this person to be served as well) and the other four wait in line. You join the end of the line. Assume that the service times are all exponential with rate 4. Denote the amount of time you will wait in line as T.

   (a) Find the mean and variance of T.

   solution: The time for each person (including the one currently being served) to be served, putting me at the front of the line, is exponential with mean 1/4 and variance 1/16. Assuming independence,

   $$ET = ES_1 + ES_2 + ES_3 + ES_4 + ES_5 = 5/4$$

   and

   $$Var[T] = Var[S_1]+Var[S_2]+Var[S_3]+Var[S_4]+Var[S_5] = 5/16$$

   (b) Write down the probability density function for T.

   Since each of these service times (S_i) is exponential, and if we assume independence, then the sum of these five exponential random variables is gamma distributed with $\alpha = 5$ and rate $\lambda = 4$. So, the density is

   $$f(x) = \frac{4^5 x^4 e^{-4x}}{\Gamma(\alpha)}$$