Math/Stat 416.2 Quiz 5 Name: ____________________

Instructions: Please read!

1. Do all work on this exam packet. It is okay to leave your answer unsimplified, as in $0.56\frac{15!}{6!9!}$ or $14e - 20$. This means that no calculator is needed.
2. Show all work for full credit! Small mistakes in arithmetic will not reduce credit if you show work; conversely, even a correct answer could get no credit without supporting work.
3. I will award partial credit where appropriate.
4. You may use all the facts from the probability review as given facts UNLESS you are explicitly asked to derive something.

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1. Suppose that there are machines A, B, and C all working at the same time and with exponential failure RATES of 1, 2, and 3 (in units of failures/hour), respectively.

   (a) Write down the density of the time until the first machine fails. solution: The time to the first failure is the minimum of the failure times which is also exponential with a rate equal to the sum of the failure times of each. So, the density of the time until the first machine fails is

   $$f(t) = 6e^{-6t}$$

   for $t > 0$

   (b) What is the probability that the first machine to fail is machine B? solution: The probability that a particular exponential random variable is the minimum among a group of exponentials is that particular exponential RV's rate over the sum of the rates.

   $$P(machine\ B) = \frac{2}{1+2+3}$$

2. Suppose that a radioactive source emits particles according to a Poisson process with a rate of 2 per minute.

   (a) What is the probability that exactly two particles are emitted in the interval from 0 to 3 minutes? solution:This is a Poisson random variable with mean equal to rate multiplied by time.

   $$P(N(3) = 2) = e^{-2*3}(2 * 3)^2/2! = e^{-6}6^2/2$$

   (b) Given that there are exactly two particles in the first 3 minutes, what is the probability that there is exactly one in the first minute? solution:This is a binomial probability as follows

   $$P(X(1) = 1|X(3) = 2) = \binom{2}{1}\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^1$$

   (c) What is the probability that the third particle arrives after 3 minutes? (You may express your answer as an integral, but you do not need to solve the integral.)

   solution: The arrival time to the third particle is gamma with $\alpha = 3$ and $\lambda = 2$. So,

   $$P(S_3 > 3) = \int_3^\infty \frac{2^3t^2e^{-2t}}{2}dt$$

3. Suppose that on a rural road, 20 trucks and 30 cars pass per hour at a certain point on that road each according to an independent Poisson process.

   (a) What is the average time until the second vehicle arrives? What is the probability that the vehicle is a truck?

   solution: The combined process for vehicles is Poisson with rate 50. So, the average time to the arrival of the second vehicle is 2/50. The probability that it is a truck is 20/50=2/5.

   (b) Given that 40 vehicles have passed by in an hour, what is the probability that there were exactly 5 trucks and 35 cars?

   solution: Since we know that there are 40 vehcicles, the probability of exactly 5 trucks and 35 cars is Binomial

$$P(5\ trucks|40\ vehicles) = \binom{40}{5}\left(\frac{2}{5}\right)^5\left(\frac{3}{5}\right)^{35}$$

4. Ellen catches fish at times of a Poisson process with rate 2 per hour.

   (a) If 40% of the fish are salmon and 60% are trout, what is the probability she will catch exactly 1 salmon and 2 trout if she fishes for 2.5 hours.

   solution: Call the Poisson process with rate 0.4(2) $N_1(t)$, the number of salmon by time t, and the independent Poisson process with rate 0.6(2) $N_(t)$, the number of trout by time t.

   $$P(N_1(2.5) = 1, N_2(2.5) = 2) = P(N_1(2.5) = 1)P(N_2(2.5) = 2)$$

   and this equals

   $$\frac{e^{-2.5(0.8)}(2.5(0.8))}{1} \frac{e^{-2.5*(1.2)}(2.5(1.2))^2}{2}$$

   (b) Ellen will fish until she gets tired. Suppose that she gets tired according to a uniform distribution from 0 to 2 hours. Find the variance of the number of fish she catches by the time she gets tired.

   solution: Let U be a uniform random variable from 0 to 2 and $N(t)$ the Poisson process representing the fish up to time t with rate 2.

   $$Var[N(U)] = E[Var[N(U)|U] + Var[E[N(U)|U]] = E[\lambda U] + Var[\lambda U]$$

   So, this is

   $$2(1/2) + 4(4/12)$$