Math/Stat 416.2 Quiz 6 Name: ____________________

Instructions: Please read!

1. Do all work on this exam packet. It is okay to leave your answer unsimplified, as in $0.56\frac{15!}{6!9!}$ or $14e - 20$. This means that no calculator is needed.
2. Show all work for full credit! Small mistakes in arithmetic will not reduce credit if you show work; conversely, even a correct answer could get no credit without supporting work.
3. I will award partial credit where appropriate.
4. You may use all the facts from the probability review as given facts UNLESS you are explicitly asked to derive something.

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1. There are two tennis courts. Pairs of players arrive at rate 3 per hour and play for an exponentially distributed amount of times with mean 1 hour. If there are already two pairs of players playing, new arrivals will leave. Express the number of pairs playing at the tennis courts as a Markov chain using the departure rates (v_is) and the probabilities of transitioning given that you are leaving a particular state (the P^*_{ij}s). solution: I found it easier to first write down the rate matrix.

$$\begin{pmatrix} -3 & 3 & 0 \\ 1 & -4 & 3 \\ 0 & 2 & -2 \end{pmatrix}$$

Note that when there are two pairs playing–the rate for one of the two pairs to leave is 2. From this we obtain,

$$v_1 = 3 \quad v_2 = 4 \quad v_3 = 2$$

and

$$P^*_{0,1} = 1 \quad P^*_{1,0} = 1/4 \quad P^*_{1,2} = 3/4 \quad P^*_{2,1} = 1$$

The rest of the probabilities are zero.

2. The continuous time Markov Chain $X(t)$ has the state space $\{1, 2, 3\}$ with the rate matrix

$$Q = \begin{pmatrix} -7 & 3 & 4 \\ 5 & -6 & 1 \\ 3 & 2 & -5 \end{pmatrix}$$

(a) If you start in state 2 at time 0, write down the density for the time of leaving state 2. solution: the time to leave the state of a Markov chain is exponential with rate v_i. So, in this case, $v_2 = 6$ and the density is

$$f(t) = 6e^{-6t}$$

for $t > 0$.

(b) Suppose there is a Poisson process, $N(t)$, with rate 7. Find the probability transition matrix (P) for the discrete time Markov chain X_n such that $X_{N(t)}$ is a continuous time Markov chain with rate matrix Q.

solution: One formula is $v_i = (1 - P_{ii})\lambda$. So,

$$P_{11} = 0 \quad P_{22} = 1/7 \quad P_{33} = 2/7$$

The other formula is $P^*_{ij} = P_{ij}/(1 - P_{ii})$ and this can be used to finish the rest of the matrix.

$$P = \begin{pmatrix} 0 & 3/7 & 4/7 \\ 5/7 & 1/7 & 1/7 \\ 3/7 & 2/7 & 2/7 \end{pmatrix}$$

3. A small barbershop, operated by a single barber, as room for at most two customers. (If there is one customer, he is having his hair cut. If there are two, one is having his hair cut, while the other waits.) Potential customers arrive according to Poisson process with a rate of three per hour, and the successive service times are independent exponential random variables with mean 1/4 hour. Find the Q matrix for continuous time Markov chain representing the number of customers in the barbershop.

$$P = \begin{pmatrix} -3 & 3 & 0 \\ 4 & -7 & 3 \\ 0 & 4 & -4 \end{pmatrix}$$

4. An insurance company pays out claims on the life insurance policies in accordance with a Poisson process having rate $\lambda = 5$ per week. The amount of money paid on the ith policy (X_i) is exponentially distributed with mean \$2,000.

   (a) Find the mean amount of money paid by the insurance company in a four-week span. solution: The amount of money in a four-week span is

$$\sum_{i=1}^{N(4)} X_i$$

   To find the mean, we condition on $N(4)$.

$$E\left[E\left[\sum_{i=1}^{N(4)} X_i \middle| N(4)\right]\right] = E\left[N(4)E[X_i]\right] = 2000E\left[N(4)\right] = 2000 \cdot 4 \cdot 5$$

   (b) Find the variance of the amount of money paid by the insurance company in a four-week span.
   solution: We again condition on $N(4)$.

$$Var\left[\sum_{i=1}^{N(4)} X_i\right] = E\left[Var\left[\sum_{i=1}^{N(4)} X_i \middle| N(4)\right]\right] + Var\left[E\left[\sum_{i=1}^{N(4)} X_i \middle| N(4)\right]\right]$$

   which we can then use our knowledge about means/variances of sums of random variables to write (Note that $Var(X_i) = (2000)^2$ because X_i is exponential.)

$$Var\left[\sum_{i=1}^{N(4)} X_i\right] = E\left[N(4)Var[X_i]\right] + Var\left[N(4)E[X_i]\right]$$

   Filling in values for the quantities of X_i we obtain

$$Var\left[\sum_{i=1}^{N(4)} X_i\right] = E\left[N(4)(2000)^2\right] + Var\left[N(4)2000\right]$$

   which leads to

$$Var\left[\sum_{i=1}^{N(4)} X_i\right] = (2000)^2 E\left[N(4)\right] + (2000)^2 Var\left[N(4)\right]$$

   Now, we use facts about the Poisson process to finish.

$$Var\left[\sum_{i=1}^{N(4)} X_i\right] = (2000)^2 \cdot 4 \cdot 5 + (2000)^2 \cdot 4 \cdot 5$$