Math/Stat 416.2 Quiz ~~6~~ 7 Name: ____________

Instructions: Please read!

1. Do all work on this exam packet. It is okay to leave your answer unsimplified, as in $0.56\frac{15!}{6!9!}$ or $14e - 20$. This means that no calculator is needed.
2. Show all work for full credit! Small mistakes in arithmetic will not reduce credit if you show work; conversely, even a correct answer could get no credit without supporting work.
3. I will award partial credit where appropriate.
4. You may use all the facts from the probability review as given facts UNLESS you are explicitly asked to derive something.

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1. (10 pts) Customers arrive at the Shortstop convenience store at a rate of 20 per hour. When two or fewer customers are present in the checkout line, a single clerk works and the average service time is 3 minutes. However, when there are three or more customers present, an assistant comes over to bag up the groceries and reduces the average service time to 2 minutes. When there are five customers in line, then people who arrive leave immediately.

   (a) Write down the rate matrix Q for an appropriate continuous time Markov chain (in units of people per hour).

$$Q = \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{pmatrix} -20 & 20 & 0 & 0 & 0 & 0 \\ 20 & -40 & 20 & 0 & 0 & 0 \\ 0 & 20 & -40 & 20 & 0 & 0 \\ 0 & 0 & 30 & -50 & 20 & 0 \\ 0 & 0 & 0 & 30 & -50 & 20 \\ 0 & 0 & 0 & 0 & 30 & -30 \end{pmatrix}$$

   (b) Write down the v_i and the $P^*_{i,j}$ for this model.

$v_0 = 20$

$v_1 = 40$

$v_2 = 40$

$v_3 = 50$

$v_4 = 50$

$v_5 = 30$

$P^*_{01} = 1$

$P^*_{10} = P^*_{12} = \frac{1}{2}$

$P^*_{21} = P^*_{23} = 1/2$

$P^*_{32} = \frac{3}{5}$ $\quad P^*_{34} = \frac{2}{5}$ $\quad P^*_{54} = 1$

$P^*_{43} = \frac{3}{5}$ $\quad P^*_{45} = \frac{2}{5}$

2. Consider a barber shop with two barbers but nowhere to wait. Customers arrive at the rate of three per hour, but if both barbers are busy, the customer leaves. EACH barber cuts hair according to an exponential random variable with an average time of 15 minutes.

   (a) Write down the Q matrix for the number of customers in the shop.

   $X(t) = \#$ of customers in shop

$$Q = \begin{array}{c} 0 \\ 1 \\ 2 \end{array} \begin{pmatrix} -3 & 3 & 0 \\ 4 & -7 & 3 \\ 0 & 8 & -8 \end{pmatrix}$$

   (b) Find the proportion of time that there are no customers in the shop and the proportion of time when arriving customers leave because it is full.

$$0 = \pi Q$$

$$-3\pi_0 + 4\pi_1 = 0 \Rightarrow \pi_1 = \frac{3}{4}\pi_0$$

$$3\pi_1 - 8\pi_2 = 0 \Rightarrow \pi_2 = \frac{3}{8}\pi_1 = \frac{9}{32}\pi_0$$

$$\pi_0 + \pi_1 + \pi_2 = 1$$

$$\pi_0 + \frac{3}{4}\pi_0 + \frac{9}{32}\pi_0 = 1 \Rightarrow \pi_0 = \frac{1}{1+\frac{3}{4}+\frac{9}{32}} = \frac{32}{65}$$

time no customers $\pi_0 = \frac{32}{65}$

time customers leave $\pi_2 = \frac{9}{65}$

3. Suppose that a factory has two machines and two repairmen. Each repairman can only work on one broken machine at a time, and the average repair time is 30 minutes. The average time of failure for an operating machine is three hours. What is the proportion of time that both machines are broken? What is the proportion of time when both machines are fixed?

$X(t) = \#$ broken machines

$$Q = \begin{array}{c} 0 \\ 1 \\ 2 \end{array} \begin{pmatrix} -\frac{2}{3} & \frac{2}{3} & 0 \\ 2 & -7/6 & 1/3 \\ 0 & 4 & -4 \end{pmatrix}$$

$$0 = \pi Q$$

$$-\frac{2}{3}\pi_0 + 2\pi_1 = 0 \Rightarrow \pi_1 = \frac{1}{3}\pi_0$$

$$\frac{1}{3}\pi_1 - 4\pi_2 = 0 \Rightarrow \pi_2 = \frac{1}{12}\pi_1 = \frac{1}{36}\pi_0$$

$$\pi_0 + \pi_1 + \pi_2 = 1 \Rightarrow \pi_0 + \frac{1}{3}\pi_0 + \frac{1}{36}\pi_0 = 1 \Rightarrow \pi_0 = \frac{36}{49}$$

both Fixed $\rightarrow \pi_0 = \frac{36}{49}$

both broken $\rightarrow \pi_2 = \frac{1}{49}$

# of people to board bus at the time of first bus $N_p(T_B)$

4. People arrive to a bus stop according to a Poisson process with rate λ (people per hour). Busses arrive according to a Poisson process (independent of the arrival of people) at the rate of μ (busses per hour).

where $T_B \sim \exp(\mu)$

N_p is Poisson process with param. λ

(a) Find the mean number of people waiting at the time of the arrival of the first bus.

$$E\, N_p(T_B) = E\left[E[N_p(T_B) \mid T_B]\right]$$
$$= E[\lambda T_B]$$
$$= \lambda E[T_B] = \frac{\lambda}{\mu}$$

(b) Find the variance of the number of people waiting at the time of the arrival of the first bus.

$$Var[N_p(T_B)] = E[Var(N(T_B) \mid T_B)] + Var[E[N(T_B) \mid T_B]]$$
$$= E[\lambda T_B] + Var[\lambda T_B]$$
$$= \lambda E[T_B] + \lambda^2 Var[T_B]$$
$$= \frac{\lambda}{\mu} + \frac{\lambda^2}{\mu^2}$$