Math/Stat 416 Fall 2015
Some Review Questions

December 3, 2015

These are (practically) all of the questions given in a previous semester's set of quizzes. They are to help you study and cover the breadth of the course. However, they are not fully comprehensive. You should also study homework questions, your own quizzes, and course notes!

1. A coin that comes up heads with probability p is continually flipped until the pattern H, H, T appears. (That is, you stop flipping when the most recent flip lands tails and the two immediately preceding it lands heads.) Let X denote the number of flips made. Find $E[X]$. You may leave your answer in terms of an equation in $E[X]$ and p.

2. Suppose that X is an exponential random variable with mean 3. Further suppose that the conditional density of Y given $X = x$ is exponential with mean x. In other words,

$$f_{Y|X}(y|x) = \frac{1}{x}e^{-y/x}$$

   for $x, y > 0$. Find the variance of Y.

3. A Markov chain with state space $\{1, 2, 3, 4, 5, 6\}$ has probability transition matrix P of the following form:

$$\begin{pmatrix} 1/3 & 0 & 2/3 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 3/4 & 0 & 0 \\ 2/3 & 0 & 1/3 & 0 & 0 & 0 \\ 0 & 1/5 & 0 & 4/5 & 0 & 0 \\ 1/4 & 1/4 & 0 & 0 & 1/4 & 1/4 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \end{pmatrix}$$

   Find all communication classes, which classes are transient/recurrent, and whether each class is periodic/aperiodc.

4. Recall the Markov chain we discussed in class concerning how many machines are broken. The state space is $\{1, 2, 3\}$. The probability transition matrix is

$$\begin{pmatrix} 0.95 & 0.05 & 0 \\ 0 & 0.9 & 0.1 \\ 1 & 0 & 0 \end{pmatrix}$$

   Further assume that the probability mass function for X_0 is given as (0.25 0.25 0.5). Find EX_2.

5. (An urn initially contains four balls that can be green or red. A ball is drawn from the urn and replaced with a ball of the opposite color. (e.g. If red is drawn, then green is returned to the urn.) If we denote the number of red balls in the urn as X_n for draw n, then describe the transition probabilities for this model. In other words, write down probability transition matrix.

6. In a good weather year, the number of storms is Poisson distributed with mean 1; in a bad year, it is Poisson distributed with mean 3. Suppose that any year's weather conditions depends on past years on through the previous year's conditions. Suppose that a good year is equally likely to be followed by either a good or bad year, and that a bad year is twice as likely to be followed by a bad year as by a good year. Suppose that last year–call it year 0–was a good year. Find the expected total number of storms in the next two years (that is in years 1 and 2.

7. The state space is $\{1, 2\}$. The probability transition matrix is

$$\begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}$$

   where α and β are both strictly between 0 and 1. Does a limiting distribution exist? If so, find it.

8. Every time that the team wins a game, it wins its next game with probability 0.8; every time it loses a game it wins its next game with probability 0.3. Assume that the team wins the third game, what is the probability that they win the fifth game?

9. A Markov chain with state space $\{1, 2, 3, 4\}$ has probability transition

matrix P of the following form:

$$\begin{pmatrix} 1/3 & 0 & 2/3 & 0 \\ 1/4 & 1/4 & 0 & 1/2 \\ 0 & 0 & 1/3 & 2/3 \\ 0 & 0 & 1/5 & 4/5 \end{pmatrix}$$

Find all communication classes, which classes are transient/recurrent, and whether each class is periodic/aperiodc. Does our theorem concerning the convergence of a Markov chain to the stationary distribution (i.e. Theorem 4.1) ensure that there is a limiting distribution? Is there a limiting distribution, i.e. does $P_{i,j}^n$ converge to some π_j regardless of i?

10. Find the probability generating function of a Poisson random variable X with mean λ. (I know we've been discussing the Poisson process a great deal recently, but this problem is asking about a Poisson random variable!)

11. Suppose that we have a branching process with a distribution for the offspring of one organism having a probability mass function

$$P(Z=0)=\frac{1}{8} \quad P(Z=1)=\frac{3}{8} \quad P(Z=2)=\frac{1}{2}$$

Do the following:

(a) Find the probability generating function for Z.

(b) What is the mean of the branching process at time $n = 100$? (Without a calculator, you will not be able to give a single number as an answer. So, report an expression that would be easily calculable if you did have a calculator.)

(c) Find the probability of extinction.

12. Suppose that we have fixed times $0 < t_1 < t_2 < t_3 < t_4$. For a Poisson process with intensity λ, find the covariance of $N(t_3) - N(t_1)$ and $N(t_4) - N(t_2)$.

13. Suppose that $N(t)$ is a Poisson process with intensity λ, and U is an independent uniform random variable between 0 and 1. Find the probability

$$P(N(U) = 0)$$

14. $N(t)$ is a Poisson process with intensity λ. Find the covariance between $N(3) - N(1)$ and $N(4) - N(2)$.

15. Suppose that $N(t)$ is a Poisson process with intensity λ. Find the following:

    (a) $P[N(2) = 3 | N(5) = 5]$

    (b) $E[N(2) | N(5) = 5]$

16. Suppose customers arrive to a ticket window according to a Poisson process with intensity/rate of $\lambda = 7$ per hour.

    (a) What is the variance of the time when the fourth customer arrives?

    (b) Assume that 40 percent of the customers are men and 60 percent are women (and that whether each customer is a man/woman is independent of every other customer). What is the expected time when the fifth woman arrives?

17. Suppose that $N(t)$ is a non-homogeneous Poisson process with intensity function $\lambda(t) = 3t^2 + 2t$. Find $E[N(3) - N(2)]$.

18. Suppose we have a discrete-time Markov chain with probability transition matrix

$$P = \begin{pmatrix} 0.25 & 0.25 & 0.5 \\ 0.3 & 0.6 & 0.1 \\ 0.2 & 0.5 & 0.3 \end{pmatrix}$$

    and $N(t)$ is a Poisson process with intensity 4. Find the distribution of the continuous time Markov chain $X_{N(t)}$ by

    (a) writing down the Q matrix,

    (b) and also writing down the v_i and P^*_{ij} as we defined them in class.

19. Define the continuous time Markov chain $X(t)$ which takes values 0 or 1 by rate matrix

$$Q = \begin{pmatrix} -3 & 3 \\ 4 & -4 \end{pmatrix}$$

    Suppose that the chain starts in state 0 at time $t = 0$. Define T as the time for the chain to leave state 0, move to state 1, and then return to state 0.

(a) Find the expected value of T.

(b) Find the variance of T.

20. A small barbershop, operated by a single barber, has room for at most two customers. Potential customers arrive at a Poisson rate of four per hour, and the successive service times are independent exponential random variables with mean 1/2 hour.

    (a) Write down the rate matrix Q for this model.

    (b) Write down the holding rates (v_i) and the transitioning probabilities given that you leave a state (P^*_{ij}) for this model.

    (c) Write down the system of three Kolmogorov forward equations, i.e. multiply things out; do not write as a matrix equation.

    (d) Write down the system of three Kolmogorov backward equations in the same way as the previous equations.

21. Suppose that $N(t)$ is a Poisson process with intensity λ and Z is an exponential random variable with intensity μ. In other words, the density of Z is

$$f(z) = \mu e^{-\mu z}$$

Find

$$P(N(Z) = 0)$$